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COMPARATIVE ANALYSIS OF A NEW INTEGRATION METHOD WITH CERTAIN STANDARD METHODS

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Recently, experiments have been made with a new numerical scheme for integrating the primitive equations. The new method may be expressed by reference to the wave equation

$$\frac{\partial \zeta}{\partial t} = i \omega \zeta \tag{1}$$

by writing

$$\zeta_*^{n+1} = \zeta^{n-1} + 2\Delta t \; i \; \omega \; \zeta_*^{n} \tag{2a}$$

$$\zeta^{n} = \alpha \; \zeta_*^{n} + .5(1-\alpha)(\zeta^{n-1} + \zeta_*^{n+1}) \tag{2b}$$

in which the index, n, fixes the time level and α is a fraction less than unity.

When α is set to unity, the scheme is the well-known "leapfrog" method. When α is set to zero, the method reduces to one studied by Kurihara [1] and called by him the "leapfrog-backward" method. To show this last point, (2b) may be rewritten as ($\alpha = 0$)

$$\zeta^{n+1} = .5(\zeta^{n} + \zeta^{n} + 2\Delta t \; i \; \omega \; \zeta_*^{n+1})$$

or

$$\zeta^{n+1} = \zeta^{n} + \Delta t \; i \; \omega \; \zeta_*^{n+1} \tag{2c}$$

If one defines $b = \omega\Delta t$, following Kurihara, the stability criterion for the leapfrog scheme is

$$b \leq 1$$

and for the leapfrog-backward scheme is

$$b \leq .8$$

Two other schemes have been used in numerical integrations of the primitive equations and analyzed by Kurihara. These are the Euler-backward scheme

$$\zeta_*^{n+1} = \zeta^{n} + \Delta t \; i \; \omega \; \zeta^{n} \tag{3a}$$

$$\zeta^{n+1} = \zeta^{n} + \Delta t \; i \; \omega \; \zeta_*^{n+1} \tag{3b}$$

for which the stability criterion is

$$b < 1.$$

and the "leapfrog-trapezoidal" method

$$\zeta_*^{n+1} = \zeta^{n-1} + 2\Delta t\ i\ \omega\ \zeta^n \tag{4a}$$

$$\zeta^{n+1} = \zeta^n + .5\ \Delta t\left(i\ \omega\ \zeta^n + i\ \omega\ \zeta_*^{n+1}\right) \tag{4b}$$

One may show that the general scheme (2) provides a solution, ζ_α^n, of the form

$$\zeta_\alpha^n = (1-\alpha)\zeta_{L.B.}^n + \alpha\ \zeta_L^n \tag{5}$$

where $\zeta_{L.B.}^n$ is the result of integration with the leapfrog-backward method, and ζ_L^n is the result of integration with the leapfrog method.

Now, interest has been expressed in the results to be expected with the method (2) for a variety of values of α. It should be noted that (5) does <u>not</u> necessarily imply stability of the new method whenever the criteria for the leapfrog-backward and leapfrog methods are satisfied separately. Therefore, we made calculations to solve the initial value problem,

$$\frac{\partial\zeta}{\partial t} = i\ \omega\ \zeta \tag{6}$$

$$\zeta \text{ at } t = 0 \text{ is } \hat{\zeta} = 1 + 0\ i, \tag{7}$$

with each of the methods discussed above. The starting procedure for use with method (2) was

$$\zeta_*^1 = \hat{\zeta} + i\ \omega\ \Delta t\ \hat{\zeta} \tag{8a}$$

$$\zeta^0 = \alpha\ \hat{\zeta} + .5(1-\alpha)(\hat{\zeta}+\zeta_*^1) \tag{8b}$$

We defined

$$R = \frac{2\pi}{\omega\Delta t}$$

which implies that the period of the wave is R intervals of time measured in Δt-units. The amplitude of the solution after 15 steps is tabulated below for various values of R and α:

α \ R	6	8	10	12	14	16	18	20	50	100
1.	>100.	1.22	1.16	1.03	1.07	1.08	1.02	1.00	1.00	1.00
.999	>100.	1.21	1.15	1.03	1.07	1.08	1.02	1.00	1.00	1.00
.990	>100.	1.15	1.12	1.03	1.06	1.07	1.02	1.00	1.00	1.00
.900	>100.	.97	.94	.99	.97	1.00	1.00	.99	1.00	1.00
.75	>100.	.56	.71	.79	.84	.88	.90	.92	.99	1.00
.50	>100.	2.71	.32	.49	.61	.69	.75	.79	.97	.99
.25	>100.	41.00	.11	.21	.36	.48	.57	.64	.94	.99
0.0	>100.	>100.	4.42	.11	.12	.25	.36	.45	.89	.97
-.25	>100.	>100.	35.44	2.74	.34	.06	.12	.22	.83	.96
E.B.	2.13	.13	.13	.19	.27	.35	.43	.50	.89	.97
L.T.	.23	.56	.75	.86	.91	.94	.96	.97	1.00	1.00

It will be noted that the leapfrog method yields amplitudes greater than unity even for $R > 2\pi$, the computational stability criterion corresponding to $b \leq 1$. This error is associated with the amplification produced by the "forward," starting scheme. It will be noted that that error is greatly reduced by using $\alpha = .90$. The empirical result for $\alpha = 0$, suggests that the instability with $R = 8,10$ (should be stable by Kurihara's result when $b < .8$, $R \simeq 8$) is also related to the "forward" start utilized with that method (see eqs. 8) and the greater weight attached to the amplified value of ζ^1_*.

Since both the leapfrog-trapezoidal and Euler-backward methods require the computation of two tendencies to advance the calculation, the scheme with $\alpha = .9$ or $.75$ seems to have considerable merit from an efficiency viewpoint.

REFERENCE

Kurihara, Y., (1965), "On the Use of Implicit and Iterative Methods for Time Integration of the Wave Equation," *Monthly Weather Review*, 93:1, pp 33-46.