U.S. DEPARTMENT OF COMMERCE
NATIONAL OCEANIC AND ATMOSPHERIC ADMINISTRATION
NATIONAL WEATHER SERVICE

JANUARY 1971

# AN ANALYSIS OF LINEAR COMPUTATIONAL STABILITY OF EXPLICIT AND IMPLICIT INTEGRATION SCHEMES FOR A TWO-LAYER MODEL USING SHUMAN'S σ-COORDINATE

OFFICE NOTE 49

by

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## 1. Introduction

In a previous Office Note (#45), an analysis was presented of the linear computational stability criteria for explicit and implicit integration schemes using a two-layer model in Phillips' $\sigma$-coordinate system. The purpose of the present note is to perform a similar analysis for the case in which the vertical coordinate is based on Shuman's definition of $\sigma$. In this case, the two layers are separated by a material surface, so that $\dot{\sigma}$ vanishes identically.

## 2. The Linear Equations

The system of equations governing the isentropic flow of an ideal, inviscid gas is linearized about a barotropic state of no-motion. The Earth's rotation, sphericity and topography are neglected. Slab-symmetry and infinite horizontal extent are assumed. The linear equations in a generalized vertical coordinate are

$$u_t + \phi_x + \overline{\alpha}\, p_x = 0 \tag{1}$$

$$(p_\sigma)_t + (\overline{p_\sigma})\, u_x + (\overline{p_\sigma})\, \dot{\sigma} = 0 \tag{2}$$

$$c_p T_t - \overline{\alpha}(p)_t + c_p \dot{\sigma}\, \Gamma = 0 , \tag{3}$$

$$\Gamma = \frac{\partial \overline{T}}{\partial \sigma} - \frac{\overline{\alpha}}{c_p} (\overline{p_\sigma})$$

$$\phi_\sigma + \overline{\alpha}\, p_\sigma + \overline{p}_\sigma\, \alpha = 0 \tag{4}$$

$$\overline{p}\, \alpha + \overline{\alpha}\, p = RT . \tag{5}$$

The symbols are standard; the overbar represents basic state values, and the unbarred variables are perturbation quantities. The subscripts denote differentiation with respect to the indicated independent variable, with the exception of the specific heat at constant pressure, $c_p$.

## 3. The Vertical Structure

| | | | | |
|---|---|---|---|---|
| | $\sigma_2 = 0$ | $\dot{\sigma} = 0$ | $p = 0$ | $\phi_2$ |
| $\sigma_2 = \frac{p}{p_T}$ | $\sigma_2 = \frac{1}{2}$ | $u_2,\ \alpha_2,\ T_2$ | $\bar{p}_2 = \frac{1}{2}\bar{p}_T$ | |
| | $\sigma_2 = 1$<br>$\sigma_1 = 0$ | $\dot{\sigma} = 0$ | $\bar{p} = \bar{p}_T$ | $\phi_1$ |
| $\sigma_1 = \frac{p - p_T}{p^* - p_T}$ | $\sigma_1 = \frac{1}{2}$ | $u_1,\ \alpha_1,\ T_1$ | $\bar{p}_1 = \frac{1}{2}(\bar{p}_* + \bar{p}_T)$ | |
| | $\sigma_1 = 1$ | $\dot{\sigma} = 0$ | $\bar{p} = \bar{p}_*$ | $\phi = 0$ |

We next introduce the definition of the vertical coordinate:

in the lower layer,

$$\sigma = \frac{p - p_T}{p_* - p_T} = \frac{\bar{p} - \bar{p}_T}{p_* - p_T} \tag{6}$$

and

in the upper domain,

$$\sigma = \frac{p}{p_T} = \frac{\bar{p}}{p_T} . \tag{7}$$

For notational convenience, we will define $P = p_* - p_T$, and $\bar{P} = \bar{p}_* - \bar{p}_T$. Thus

$$(p_1)_\sigma = P \tag{8}$$

and

$$(p_2)_\sigma = p_T \tag{9}$$

The equations for each layer may now be written, noting that $\dot{\sigma}$ vanishes identically everywhere.

$$(u_1)_t + \tfrac{1}{2}(\phi_1)_x + \tfrac{1}{2}\,\overline{\alpha}_1\, P_x + \overline{\alpha}_1 (p_T)_x = 0 \tag{10}$$

$$P_t + \overline{P}(u_1)_x = 0 \tag{11}$$

$$c_p (T_1)_t - \tfrac{1}{2}\,\overline{\alpha}_1\, P_t - \overline{\alpha}_1 (p_T)_t = 0 \tag{12}$$

$$-\phi_1 + \overline{\alpha}_1 P + \overline{P}\,\alpha_1 = 0 \tag{13}$$

$$\tfrac{1}{2}\,\overline{P}\,\alpha_1 + \overline{p}_T\,\alpha_1 + \tfrac{1}{2}\,\overline{\alpha}_1\, P + \overline{\alpha}_1\, p_T = RT_1 \tag{14}$$

For the upper layer,

$$(u_2)_t + \tfrac{1}{2}(\phi_1)_x + \tfrac{1}{2}(\phi_2)_x + \tfrac{1}{2}\,\overline{\alpha}_2 (p_T)_x = 0 \tag{15}$$

$$(p_T)_t + \overline{p}_T (u_2)_x = 0 \tag{16}$$

$$c_p (T_2)_t - \tfrac{1}{2}\,\overline{\alpha}_2 (p_T)_t = 0 \tag{17}$$

$$\phi_1 - \phi_2 + \overline{\alpha}_2\, p_T + \overline{p}_T\,\alpha_2 = 0 \tag{18}$$

$$\tfrac{1}{2}\,\overline{p}_T\,\alpha_2 + \tfrac{1}{2}\,\overline{\alpha}_2\, p_T = RT_2 \tag{19}$$

## 4. The Finite Difference Equations

We next introduce explicit and implicit integration schemes. It should be noted that the so-called 'modified implicit' scheme of Office Note #45 collapses to the unmodified implicit scheme here, since $\dot{\sigma}$ vanishes everywhere. The difference equations are, for the explicit scheme,

$$\frac{u_1^{n+1} - u_1^{n-1}}{2\Delta t} + \tfrac{1}{2}(\phi_1)^n_x + \tfrac{1}{2}\,\overline{\alpha}_1 (P)^n_x + \overline{\alpha}_1 (p_T)^n_x = 0 \qquad \text{E1.1}$$

$$\frac{P^{n+1} - P^{n-1}}{2\Delta t} + \bar{P}(u_1)_x^n = 0 \qquad \text{E1.2}$$

$$c_p \frac{T_1^{n+1} - T_1^{n-1}}{2\Delta t} - \tfrac{1}{2}\,\bar{\alpha}_1 \frac{P^{n+1} - P^{n-1}}{2\Delta t} - \bar{\alpha}_1 \frac{p_T^{n+1} - p_T^{n-1}}{2\Delta t} = 0 \qquad \text{E1.3}$$

$$-\phi_1^n + \bar{\alpha}_1\, P^n + \bar{P}\,\alpha_1^n = 0 \qquad \text{E1.4}$$

$$\tfrac{1}{2}\,\bar{P}\,\alpha_1^n + \bar{p}_T\,\alpha_1^n + \tfrac{1}{2}\,\bar{\alpha}_1\, P^n + \bar{\alpha}_1\, p_T^n = RT_1^n \qquad \text{E1.5}$$

$$\frac{u_2^{n+1} - u_2^{n-1}}{2\Delta t} + \tfrac{1}{2}(\phi_1)_x^n + \tfrac{1}{2}(\phi_2)_x^n + \tfrac{1}{2}\,\bar{\alpha}_2 (p_T)_x^n = 0 \qquad \text{E2.1}$$

$$\frac{p_T^{n+1} - p_T^{n-1}}{2\Delta t} + \bar{p}_T(u_2)_x^n = 0 \qquad \text{E2.2}$$

$$c_p \frac{T_2^{n+1} - T_2^{n-1}}{2\Delta t} - \tfrac{1}{2}\,\bar{\alpha}_2 \frac{p_T^{n+1} - p_T^{n-1}}{2\Delta t} = 0 \qquad \text{E2.3}$$

$$\phi_1^n - \phi_2^n + \bar{\alpha}_2\, p_T^n + \bar{p}_T\,\alpha_2^n = 0 \qquad \text{E2.4}$$

$$\bar{p}_T\,\alpha_2^n + \bar{\alpha}_2\, p_T^n = 2\,RT_2 \qquad \text{E2.5}$$

For the implicit scheme,

$$\frac{u_1^{n+1} - u_1^{n-1}}{2\Delta t} + \tfrac{1}{4}(\phi_1)_x^{n+1} + \tfrac{1}{4}(\phi_1)_x^{n-1} + \tfrac{1}{4}\,\bar{\alpha}_1\, P_x^{n+1} + \tfrac{1}{4}\,\bar{\alpha}_1\, P_x^{n-1}$$

$$+ \tfrac{1}{2}\,\bar{\alpha}_1 (p_T)_x^{n+1} + \tfrac{1}{2}\,\bar{\alpha}_1 (p_T)_x^{n-1} = 0 \qquad \text{S1.1}$$

$$\frac{P^{n+1} - P^{n-1}}{2\Delta t} + \tfrac{1}{2}\,\bar{P}(u_1)_x^{n+1} + \tfrac{1}{2}\,\bar{P}(u_1)_x^{n-1} = 0 \qquad \text{S1.2}$$

Equations S1.3 - S1.5 are the same as E1.3 - E1.5.

$$\frac{u_2^{n+1} - u_2^{n-1}}{2\Delta t} + \tfrac{1}{4}(\phi_1)_x^{n+1} + \tfrac{1}{4}(\phi_1)_x^{n-1} + \tfrac{1}{4}(\phi_2)_x^{n+1} + \tfrac{1}{4}(\phi_2)_x^{n-1}$$

$$+ \tfrac{1}{4}\,\overline{\alpha}_2 (p_T)_x^{n+1} + \tfrac{1}{4}\,\overline{\alpha}_2 (p_T)_x^{n-1} = 0 \qquad \text{S2.1}$$

$$\frac{p_T^{n+1} - p_T^{n-1}}{2\Delta t} + \tfrac{1}{2}\,\overline{p}_T (u_2)_x^{n+1} + \tfrac{1}{2}\,\overline{\overline{p}}_T (u_2)_x^{n-1} = 0 \qquad \text{S2.2}$$

Equations S2.3 - S2.5 are the same as E2.3 - E2.5.

## 5. The Characteristic Equation

We now assume solutions of the form

$$q^n = q\,\zeta^n\,e^{ikx} \qquad (20)$$

and substitute this for all dependent variables. This results in

$$u_1 + \tfrac{1}{2}\beta\,\phi_1 + \tfrac{1}{2}\,\overline{\alpha}_1\,\beta\,P + \overline{\alpha}_1\,\beta\,p_T = 0 \qquad (21)$$

$$P + \beta\,\overline{P}\,u_1 = 0 \qquad (22)$$

$$c_p\,T_1 - \tfrac{1}{2}\,\overline{\alpha}_1 P - \overline{\alpha}_1\,p_T = 0 \qquad (23)$$

$$-\phi_1 + \overline{\alpha}_1\,P + \overline{P}\,\alpha_1 = 0 \qquad (24)$$

$$\tfrac{1}{2}\,\overline{P}\,\alpha_1 + \overline{p}_T\,\alpha_1 + \tfrac{1}{2}\,\overline{\alpha}_1\,P + \overline{\alpha}_1\,p_T = RT_1 \qquad (25)$$

$$u_2 + \tfrac{1}{2}\beta\,\phi_1 + \tfrac{1}{2}\beta\,\phi_2 + \tfrac{1}{2}\beta\,\overline{\alpha}_2\,p_T = 0 \qquad (26)$$

$$p_T + \beta\,\overline{p}_T\,u_2 = 0 \qquad (27)$$

$$c_p T_2 - \tfrac{1}{2} \bar{\alpha}_2 p_T = 0 \tag{28}$$

$$\phi_1 - \phi_2 + \bar{\alpha}_2 p_T + \bar{p}_T \alpha_2 \tag{29}$$

$$\bar{p}_T \alpha_2 + \bar{\alpha}_2 p_T = 2 RT_2 \tag{30}$$

where

$$\beta = \frac{2\, ik\Delta t\, \zeta}{\zeta^2 - 1} \qquad \text{explicit}$$

$$\beta = \frac{ik\Delta t(\zeta^2 + 1)}{\zeta^2 - 1} \qquad \text{implicit} \tag{31}$$

Eqns. (21-30) can be reduced to two equations in P and $p_T$ by successive substitutions. First, $T_1$ and $T_2$ are eliminated using the pairs (23, 25) and (28, 30); this yields

$$\bar{\alpha}_1(\kappa - 1)p_1 - \bar{p}_1 \alpha_1 = 0 \tag{32a}$$

$$\bar{\alpha}_2(\kappa - 1)p_2 - \bar{p}_2 \alpha_2 = 0 \qquad (\kappa \equiv R/c_p), \tag{32b}$$

Then $\alpha_1$ and $\alpha_2$ may be eliminated between the pairs (24,32a) and (29,32b):

$$-\phi_1 + \bar{\alpha}_1 [1 + 2\, \varepsilon(\kappa - 1)\delta]P = 0 \tag{33a}$$

$$-\phi_2 + \phi_1 + \kappa\, \bar{\alpha}_2 p_T = 0 \tag{33b}$$

where $\varepsilon \equiv \dfrac{\bar{p}_* - \bar{p}_T}{\bar{p}_* + \bar{p}_T}$ and $\delta \equiv \dfrac{p_1}{p_* - p_T}$ .

Next, $u_1$ and $u_2$ may be eliminated between the pairs (21,22) and (26,27):

$$\left(1 - \beta^2\, \bar{P}\, \bar{\alpha}_1\, \delta\right)P - \tfrac{1}{2} \beta^2\, \bar{P}\, \phi_1 = 0 \tag{34a}$$

$$\left(1 - \tfrac{1}{2} \beta^2\, \bar{p}_T\, \bar{\alpha}_2\right)p_T - \tfrac{1}{2} \beta^2\, \bar{p}_T\, \phi_1 - \tfrac{1}{2} \beta^2\, \bar{p}_T\, \phi_2 = 0 \tag{34b}$$

One may then replace $\phi_1$ and $\phi_2$ as they appear in (34) from (33). After manipulation, this yields

$$\left(1 - \beta^2\varepsilon\, R\overline{T}_1[2 + \varepsilon(\kappa-1)]\right)P - \left(2\, \beta^2\varepsilon\, R\overline{T}_1[1 + \varepsilon(\kappa-1)]\right)p_T = 0 \qquad (35a)$$

$$\left(2\, \beta^2 r\, R\overline{T}_2[1 + \varepsilon(\kappa-1)]\right)P - \left(1 - \beta^2 R\overline{T}_2[1 + \kappa + 4\varepsilon r(\kappa-1)]\right)p_T = 0 \qquad (35b)$$

where $r \equiv \overline{\alpha}_1/\overline{\alpha}_2$. The determinant of (35) must vanish, which leads to the frequency equation,

$$\varepsilon(R\overline{T}_1)(R\overline{T}_2)\left([2 + \varepsilon(\kappa-1)][1 + \kappa + 4\varepsilon r(\kappa-1)] - 4r[1 + \varepsilon(\kappa-1)]^2\right)\beta^4$$
$$- \left(\varepsilon\, R\overline{T}_1[2 + \varepsilon(\kappa-1)] + R\overline{T}_2[1 + \kappa + 4\varepsilon r(\kappa-1)]\right)\beta^2 + 1 = 0 \qquad (36)$$

We next introduce a change of variables, $z \equiv (\beta c)^2$, where $c = \gamma\, R\overline{T}$, with $\gamma = c_p/c_v$. Eqn. (36) becomes a quadratic in $z$,

$$a\, z^2 + b\, z + 1 = 0 \qquad (37a)$$

where

$$a = [\gamma\, R\overline{T}]^{-2}\varepsilon(R\overline{T}_1)(R\overline{T}_2)\left([2 + \varepsilon(\kappa-1)][1 + \kappa + 4\varepsilon r(\kappa-1)] - 4r[1 + \varepsilon(\kappa-1)]^2\right) \qquad (37b)$$

and

$$b = -[\gamma\, R\overline{T}]^{-1}\left(\varepsilon\, R\overline{T}_1[2 + \varepsilon(\kappa-1)] + R\overline{T}_2[1 + \kappa + 4r\varepsilon(\kappa-1)]\right). \qquad (37c)$$

Here $\overline{T}$ is the mean temperature of the fluid; i.e., $\overline{T} = \frac{1}{2}(\overline{T}_1 + \overline{T}_2)$.

## 6. The Isothermal Atmosphere

We now seek to determine the stability criterion and the free modes allowed in this model for a particular basic state. It will be assumed that the basic state is isothermal at temperature $\overline{T}$, and that the material surface is at 500 mb:

$$\overline{T}_1 = \overline{T}_2 = \overline{T} = 250\text{K}$$
$$\overline{p}^* = 1000 \text{ mb}$$
$$\overline{p}_T = 500 \text{ mb}$$
$$\kappa = 2/7$$

From these values, we may calculate

$$\varepsilon = 2/3$$

and

$$r = 1/3.$$

The coefficients a and b of (37a) may then be calculated, and the roots $z_j$ of (37a) obtained. Proceeding as in Office Note #45, we may use the $z_j$ to investigate the stability criterion for the two integration schemes. From the definition of $\beta$ for the explicit case, we have

$$z_j \equiv - c^2 \frac{4(k\Delta t)^2 \zeta^2}{(\zeta^2 - 1)^2} \quad . \tag{38}$$

We define the ratio $\frac{c^2}{z_j}$ as the critical phase speed for stability

$$c_j \equiv \frac{c^2}{z_j} \tag{39}$$

so that (38) becomes

$$\zeta^2 - 1 = \pm i(2\ k\Delta t\ c_j)\zeta \tag{40}$$

We may solve (40) for $\zeta$ to obtain

$$\zeta = \pm\ i(k\Delta t\ c_j) \pm [1 - (k\Delta t\ c_j)^2]^{\frac{1}{2}} \tag{41}$$

so that if $(k\Delta t\ c_j) < 1$, $|\zeta| < 1$, and the explicit method will be neutral.

For the implicit case,

$$- (k\Delta t)^2 [\ \frac{\zeta^2 + 1}{\zeta^2 - 1}\ ]^2\ \frac{c^2}{z_j} = 1 \tag{42}$$

which yields the quadratics

$$\zeta^2 = \frac{c_j k\Delta t + i}{-c_j k\Delta t + i} \tag{43}$$

and

$$\zeta^2 = \frac{i - c_j k\Delta t}{i + c_j k\Delta t} \tag{44}$$

In both cases $|\zeta^2| = 1$, so that $|\zeta| = 1$. The implicit is therefore unconditionally neutral.

Finally, we may calculate the critical phase speeds $c_j$ from (41) and the roots $z_j$ of (37a);

$$z_1 = 1.06 \tag{45}$$

$$z_2 = 5.96 \tag{46}$$

The critical phase speeds are

$$c_1 = 308.0 \text{ m sec}^{-1} \tag{47}$$

$$c_2 = 130.1 \text{ m sec}^{-1} \tag{48}$$

From Office Note #45, the corresponding values for the case of the Phillips' σ-coordinate are

$$c_1 = 307.5 \text{ m sec}^{-1}* \tag{49}$$

$$c_2 = 83.3 \text{ m sec}^{-1}* \tag{50}$$

The fundamental mode, represented by $c_1$, is thus seen to be insensitive to the presence or absence of a material surface separating the upper and lower layers of the fluid. However, the phase speed of the secondary mode in the present case exceeds that of the corresponding mode in the Phillips' coordinate case by nearly 50 m $\text{sec}^{-1}$. The interpretation of this behavior is not completely clear, but it appears that the secondary mode in the present case, where $\dot{\sigma}$ vanishes at the interface, is closely akin to a free-surface mode, whereas in the case of the Phillips' σ-coordinate, the secondary mode is clearly of an internal type.

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* Subsequent to the publication of Office Note #45, an arithmetical error was discovered in the evaluation of the roots $z_j$. The root $z_1$ (eqn. 32, p.6) should be 2.96 rather than 3.9. The coefficient of $\gamma\, R\overline{T}$) in eqn. 39, p.7, then becomes 0.068 rather than 0.05.