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An Iterative Variational Method for Adjusting Isobaric Wind and Geopotential to Satisfy the Balance Equation

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1. Introduction

If the initial wind and geopotential fields are sufficiently out-of-balance, a primitive equation, prediction model will produce a gravitational-inertial oscillation as a result of the implied need for a "geostrophic adjustment." To reduce the amplitude of such oscillations, one may seek to impose the "balance-equation" upon the initial data.

The balance-equation may be used to calculate the wind from the geopotential, or vice versa. In either direction, one may find that the changes in the initial data are somewhat excessive.

In this note, we discuss a variational method for adjusting <u>both</u> the wind and the geopotential so that the final fields satisfy the balance-equation. The mutual adjustment is carried out so as to minimize the areal average of the change in actual and geostrophic kinetic energy. By allowing both the wind and the geopotential to be modified, one hopes to minimize the departures of both fields from the initial analysis.

2. The Formulation of the Method

In isobaric coordinates, the balance equation may be written

$$\nabla^2\phi - f_o\nabla^2\psi - \nabla\cdot(\eta - f_o)\nabla\psi + \tfrac{1}{2}\nabla^2(\nabla\psi\cdot\nabla\psi) = 0, \qquad (1)$$

in which f_o is a constant (10^{-4} sec^{-1}), η is the absolute vorticity, ϕ is the absolute vorticity, ϕ is the geopotential, and ψ is the streamfunction.

Denote by A the ageostrophic terms,

$$A \equiv - \nabla\cdot(\eta - f_o)\nabla\psi + \tfrac{1}{2}\nabla^2(\nabla\psi\cdot\nabla\psi). \qquad (2)$$

Now let the analysis yield initial values ψ^* and ϕ^* for the streamfunction and geopotential. When these are introduced into (1) (with (2)), we may find that,

$$\nabla^2\phi^* - f_o\nabla^2\psi^* + A^* \equiv N^* \neq 0 \qquad (3)$$

This means that the initial data are "out-of-balance."

Suppose that we wish to adjust the analysis values so as to obtain a balanced set of streamfunction and geopotential. Let ψ_c and ϕ_c denote changes in the analyzed values, and define ψ' and ϕ' by

$$\phi' = \phi^* + \phi_c \qquad (4a)$$

$$\psi' = \psi^* + \psi_c \qquad (4b)$$

We consider the use of an iterative procedure for effecting a final balance. As a first step we try to find ψ_c, ϕ_c, such that

$$\nabla^2\phi' - f_o\nabla^2\psi' + A^* = 0, \tag{5}$$

and such that

$$I \equiv \iint_D [(\nabla\phi_c)^2 + (f_o\nabla\psi_c)^2]dxdy \tag{6}$$

is a minimum (where D is the entire domain).

From (3) and (5), we get

$$\nabla^2\phi_c - f_c\nabla^2\psi_c + N^* = 0 \tag{7}$$

Since the minimization of (6) is subject to the constraint that (7) be satisfied, we use the Lagrance multiplier λ (a function of position, but independent of ψ_c and ϕ_c) to involve the constraint (7) into the integral (6):

$$I = \iint[(\nabla\phi_c)^2 + (f_o\nabla\psi_c)^2 + \lambda[\nabla^2\phi_c - f_o\nabla^2\psi_c + N^*]]dxdy. \tag{8}$$

We consider the variation of I, δI, caused by: a variation in ϕ_c, $\delta\phi_c$, a variation of ψ_c, $\delta\psi_c$; and a variation of λ, $\delta\lambda$. It is necessary that δI vanish if I is to be a minimum.

$$\delta I = \frac{\partial I}{\partial(\delta\phi_c)}\delta\phi_c + \frac{\partial I}{\partial(\delta\psi_c)}\partial\psi_c + \frac{\partial I}{\partial(\delta\lambda)}\delta\lambda = 0. \tag{9}$$

Since each of the variations $\delta\phi_c$, $\delta\psi_c$ and $\delta\lambda$ is arbitrary and independent, it is necessary that

$$\frac{\partial I}{\partial(\delta\phi_c)} = 0 \tag{10a}$$

$$\frac{\partial I}{\partial(\delta\psi_c)} = 0 \tag{10b}$$

$$\frac{\partial I}{\partial(\delta\lambda)} = 0\ . \tag{10c}$$

To evaluate the partial derivatives, one proceeds as follows:

$$\frac{\partial I}{\partial(\delta\phi_c)} = \lim_{\delta\phi_c\to o}\left(\frac{I(\phi_c + \delta\phi_c,\ \psi_c,\ \lambda) - I(\phi_c,\ \psi_c,\ \lambda)}{\delta\phi_c}\right)_{\psi_c,\lambda \text{ consts.}} \tag{11}$$

$$I(\phi_c+\delta\phi_c,\psi_c,\lambda) = \iint(\nabla\phi_c+\nabla\delta\phi_c)^2 + (f_o\nabla\psi_c)^2 + \lambda[\nabla^2\phi_c+\nabla^2\delta\phi_c-f_o\nabla^2\psi_c+N^*]dxdy. \quad (12)$$

So

$$I(\phi_c+\delta\phi_c,\psi_c,\lambda) - I(\phi_c,\psi_c,\lambda) = \iint[2\nabla\phi_c\cdot\nabla\delta\phi_c + (\nabla\delta\phi_c)^2 + \lambda\nabla^2\delta\phi_c]dxdy. \quad (13)$$

Now if we consider the limiting ratio (11), the term

$$\iint(\nabla\delta\phi_c)^2dxdy$$

may be neglected as being of higher order in the small quantity $\delta\phi_c$. Thus one is led to require that

$$\iint[2\nabla\phi_c\cdot\nabla\delta\phi_c + \lambda\nabla^2\delta\phi_c]dxdy = 0 \quad (14)$$

Similarly one has,

$$\iint[2f_o^2\nabla\psi_c\cdot\nabla\delta\psi_c - \lambda\nabla^2\delta\psi_c]dxdy = 0 \quad (15)$$

and the constraint equation,

$$\nabla^2\phi_c - f_o\nabla^2\psi_c + N^* = 0. \quad (16)$$

By invoking the identities,

$$\nabla\psi_c\cdot\nabla\delta\psi_c = \nabla\cdot(\delta\psi_c\ \nabla\psi_c) - \delta\psi_c\ \nabla^2\psi_c$$

$$\lambda\nabla^2\delta\psi_c = \nabla\cdot(\lambda\nabla\delta\psi_c - \delta\psi_c\nabla\lambda) + \delta\psi_c\nabla^2\lambda$$

$$\nabla\phi_c\cdot\nabla\delta\phi_c = \nabla\cdot(\delta\phi_c\ \nabla\phi_c) - \delta\phi_c\ \nabla^2\phi_c$$

$$\lambda\nabla^2\delta\phi_c = \nabla\cdot(\lambda\nabla\delta\phi_c - \delta\phi_c\nabla\lambda) + \delta\phi_c\nabla^2\lambda$$

together with Gauss' theorem, one may rewrite (14) and (15) as follows:

$$\oint\{\delta\phi_c[2\vec{n}\cdot\nabla\phi_c - \vec{n}\cdot\nabla\lambda] + \lambda\vec{n}\cdot\nabla\delta\phi_c\}d\ell + \iint \delta\phi_c[-2\nabla^2\phi_c + \nabla^2\lambda]dxdy = 0 \quad (14a)$$

$$\oint\{\delta\psi_c[2f_o\vec{n}\cdot\nabla\psi_c + \vec{n}\cdot\nabla\lambda] - f_o\lambda\vec{n}\cdot\nabla\psi_c\}d\ell + \iint - \delta\psi_c[2f_o^2\nabla^2\lambda + f_o\nabla^2\lambda]dxdy = 0 \quad (15a)$$

The line integrals in 14a and 15a are carried out over the boundary curve enclosing the two-dimensional surface. The unit vector $\vec{n}$ is the outward normal on the boundary curve.

For (14a) and (14b) to be satisfied for arbitrary values of $\delta\psi_c$ and $\delta\phi_c$, it is sufficient that

$$\lambda = 0$$

$$2\vec{n}\cdot\nabla\phi_c = \vec{n}\cdot\nabla\lambda \qquad (17)$$

$$f_o 2\vec{n}\cdot\nabla\psi_c = -\vec{n}\cdot\nabla\lambda$$

on the boundary curve, and that in the interior

$$-2\nabla^2\phi_c + \nabla^2\lambda = 0 \qquad (18)$$

$$2f_o\nabla^2\psi_c + \nabla^2\lambda = 0 \,. \qquad (19)$$

Adding (18) and (19) and using (16), one has

$$\nabla^2\lambda + N^* = 0 \,. \qquad (20)$$

One may solve (20) using the homogeneous Dirichlet boundary condition on λ (see (17)). With λ known, one may satisfy (18), (19) and the remaining boundary conditions (17), by writing

$$\phi_c = \tfrac{1}{2}\lambda \qquad (21)$$

$$\psi_c = -\frac{1}{2f_o}\lambda \,. \qquad (22)$$

Having obtained ψ_c and ϕ_c, one may construct ψ' and ϕ'. The latter values may then be employed as ψ^*, ϕ^* in another iteration of the process. The iteration may be stopped when further steps produce negligible additional changes, ϕ_c, or when N^* becomes sufficiently small.

The convergence of the iterative process has not been proven. In experiments, we have found the method to converge, in the sense that the required changes in ϕ become less than 9.8 $m^2\ sec^{-2}$, after about 10 iterations.

3. The Practical Coding

In the derivation, we have used the streamfunction ψ to represent the wind field. In practice we have employed the wind components themselves to evaluate N^*. We have not removed the divergence from the analyzed wind fields, although this could be done if it were found necessary.

The computational code uses the following finite-difference formulas to evaluate N* at all interior grid points:

$$N^* = m^2 \overline{\left(g_o z + .5(u^2 + v^2)\right)}^{xx}_{yy} + m^2 \overline{\left(g_o z + .5(u^2 + v^2)\right)}^{yy}_{xx}$$

$$- f_o \overline{\zeta}^{xy} - m^2 \overline{(\overline{f}^{xy} + \zeta - f_o)\,\overline{\left(\frac{v}{m}\right)}^{xy}_{x}}^{y}$$

$$+ m^2 \overline{(\overline{f}^{xy} + \zeta - f_o)\,\overline{\left(\frac{u}{m}\right)}^{xy}}^{x}_{y}$$

with

$$\zeta = \overline{m^2}^{xy} \left[\overline{\left(\frac{v}{m}\right)_x}^{y} - \overline{\left(\frac{u}{m}\right)_y}^{x} \right]$$

and m the map factor of the polar stereographic projection true at 60°N.

The equation (18) is solved numerically using the SOR method with the "+" stencil for the Laplacian operator. The solution is iterated until λ is known to within 9.8 m^2 sec^{-2}.

Having obtained λ we calculated the revised values of geopotential height z and wind components using the formulas:

$$z = z + .5\ \lambda / g_o$$

$$u = u + .5\ f_o\ \overline{\lambda_y}^{y}$$

$$v = v - .5\ f_o\ \overline{\lambda_x}^{x}\ .$$

Note that u, v, and z are unchanged on the boundary gridpoints.

4. Results

Figures 1 shows the 300-mb height and wind speed analysis for a portion of the 65 x 65 grid. The height contours are drawn at 60 m intervals; the isotachs are at 10 m s^{-1} intervals.

Figure 2 shows the initialized fields after nine iterations. In table 1, we show the maximum absolute value of the height adjustment and the average absolute height change for each of the nine iterations.

Table 1

Iteration	Max. Abs. Change (m)	Avg. Abs. Change (m)
1	62.8	10.5
2	18.6	3.08
3	9.60	1.21
4	5.47	.53
5	3.34	.25
6	2.16	.13
7	1.46	.07
8	1.03	.04
9	.75	.02

Figure 3 shows the change in wind speed between the initialized and analyzed isotachs. The contours are at intervals of 2 m s^{-1}. The maximum changes are about 10 m s^{-1} (20 knots).

Figure 4 shows the change in the height field between the initialized and analyzed fields. The maximum change is about 70 m. The contour interval is 20 m.

Figure 5 shows the changes in the height field that would have been required, if one adjusted the analyzed height field to fit the analyzed wind field through the balance equation. The boundary values of the height field were held constant. The contours are at 60 m intervals, and the maximum change is - 125 m.

It seems to us that the initialized fields differ from the analyzed fields by amounts that are within the limits of accuracy of analyses. If this result is replicated in subsequent experiments the use of the method may be justified in operational practice.

As developed here, the method is applicable to the 65 x 65 hemispheric grid. Extension to other grids or to a spectral model appears to be tractable.

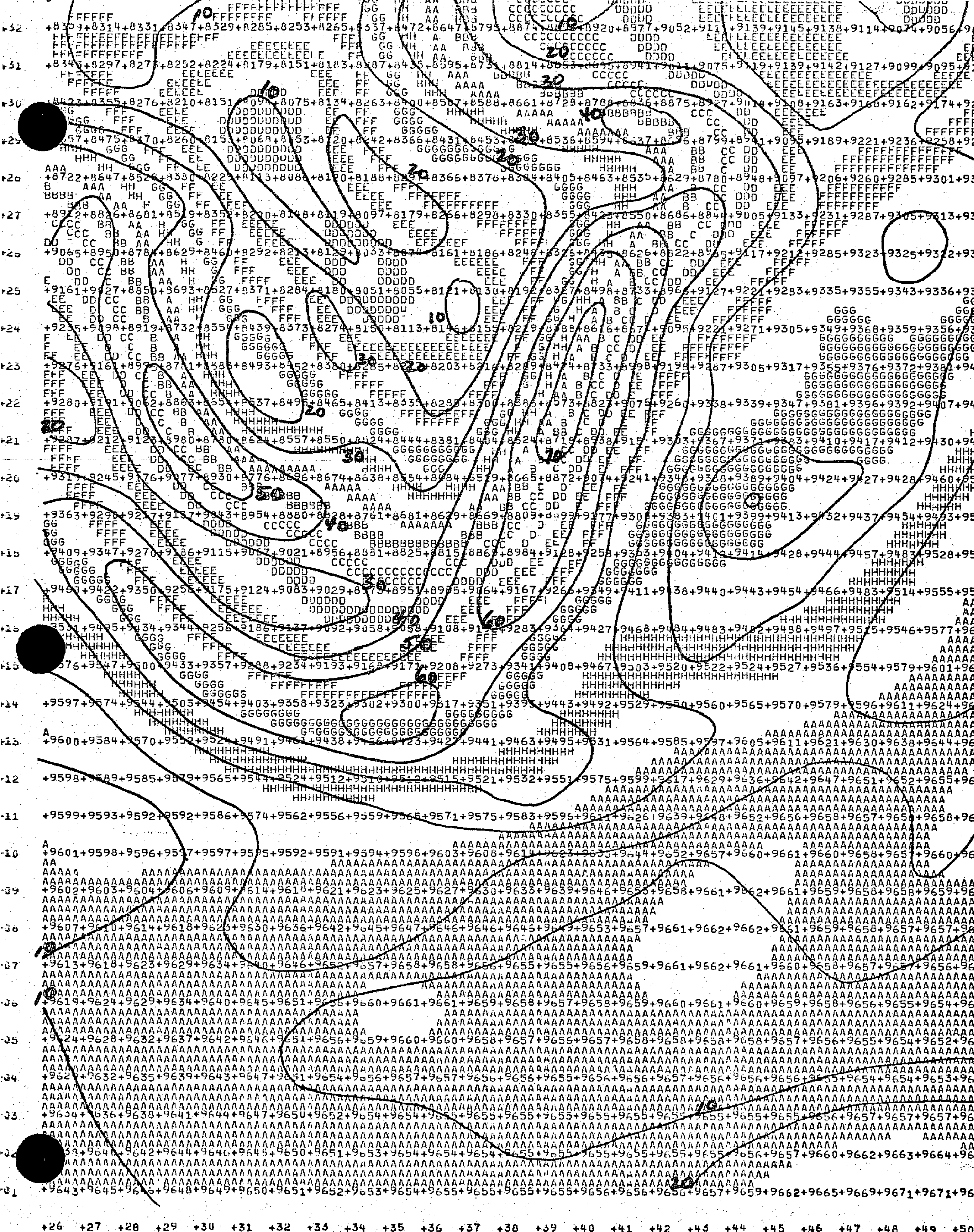


FIG 1

HGT M ANALYSIS + ISOTACHS m s^{-1}

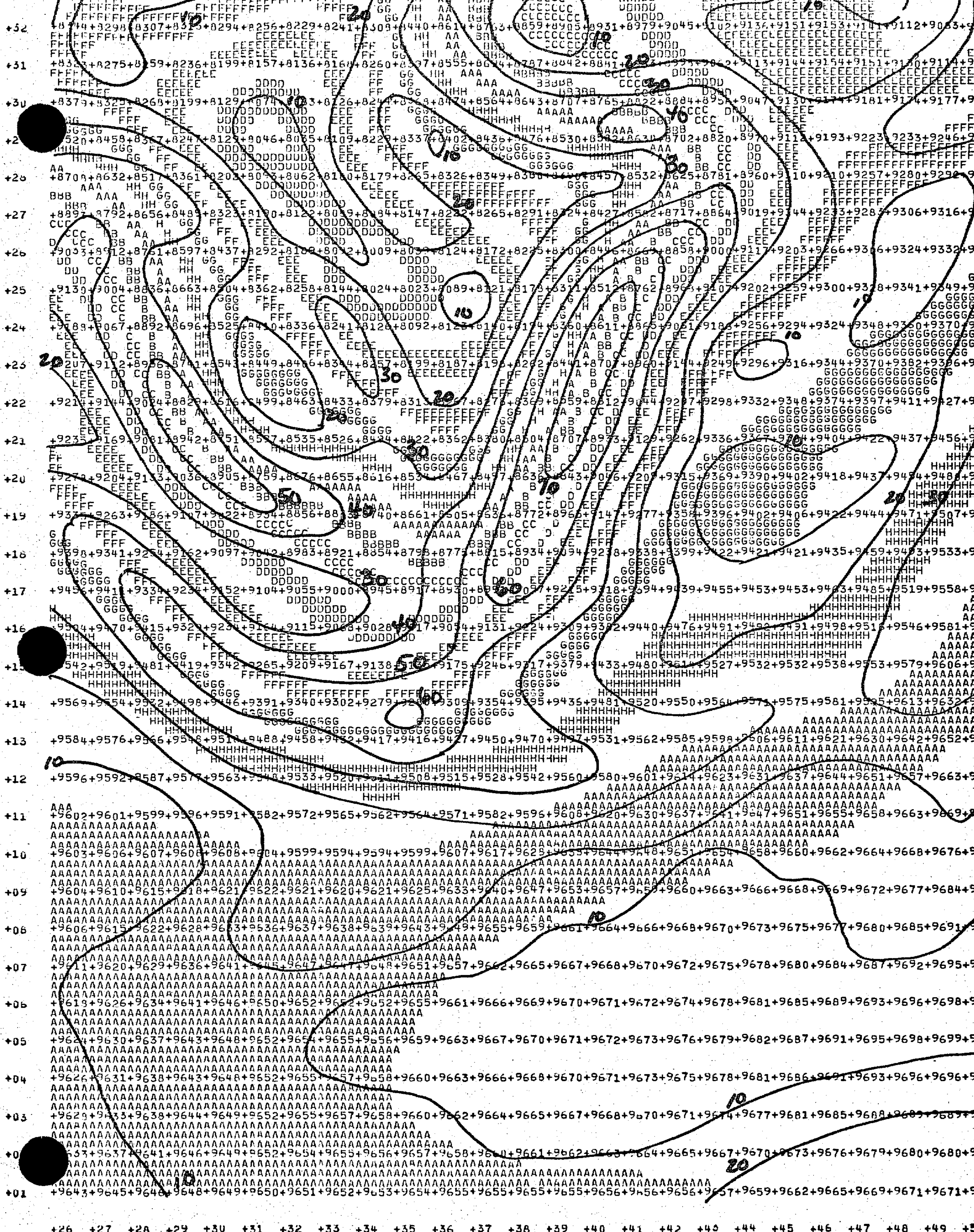
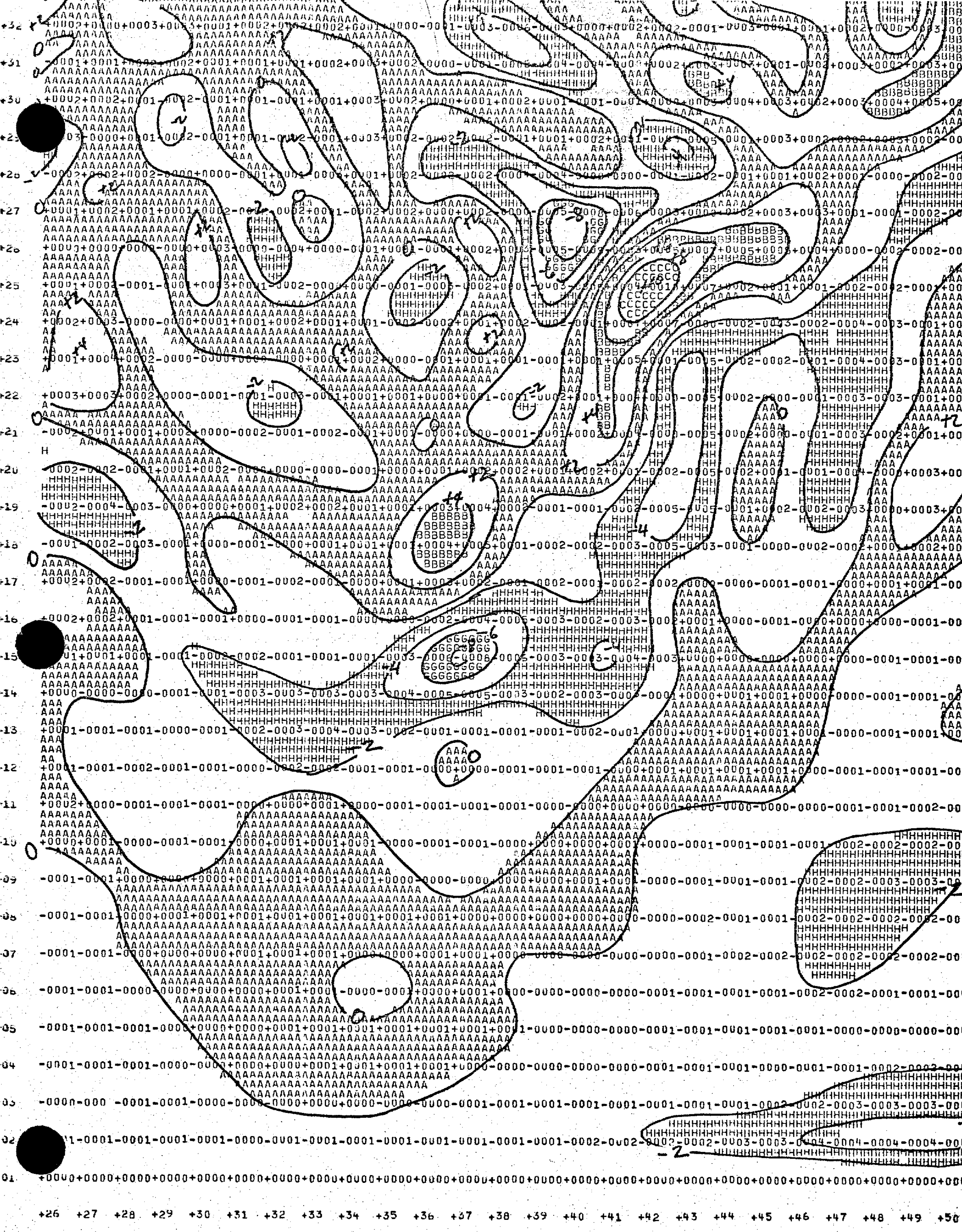
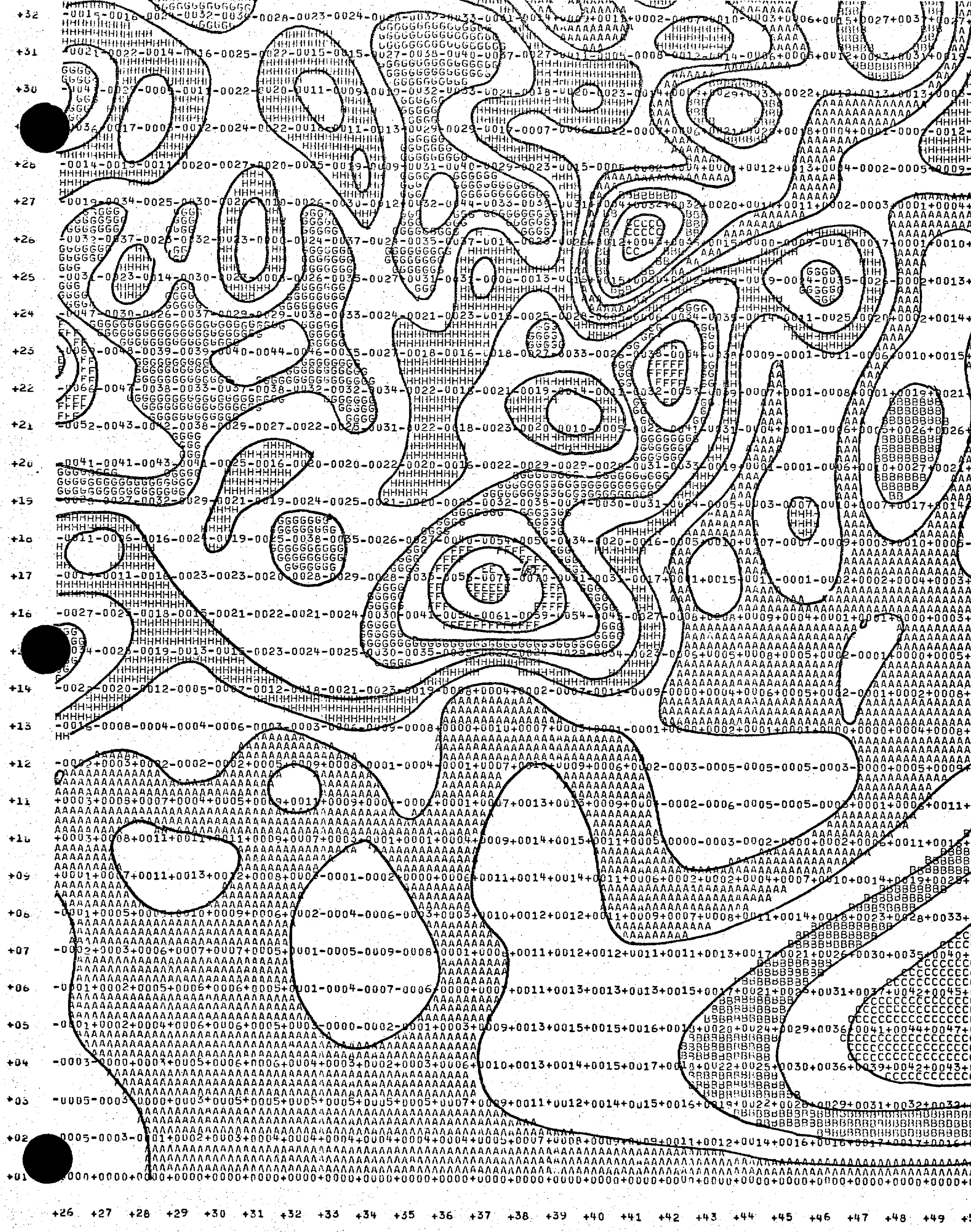


FIGURE 2

ISOTACHS & HGT M INITIALIZED



CHANGE IN SPD INITIALIZED — ANALYSIS FIG 3.



CHANGE IN HGT INITIALIZED – ANALYSIS FIG 4

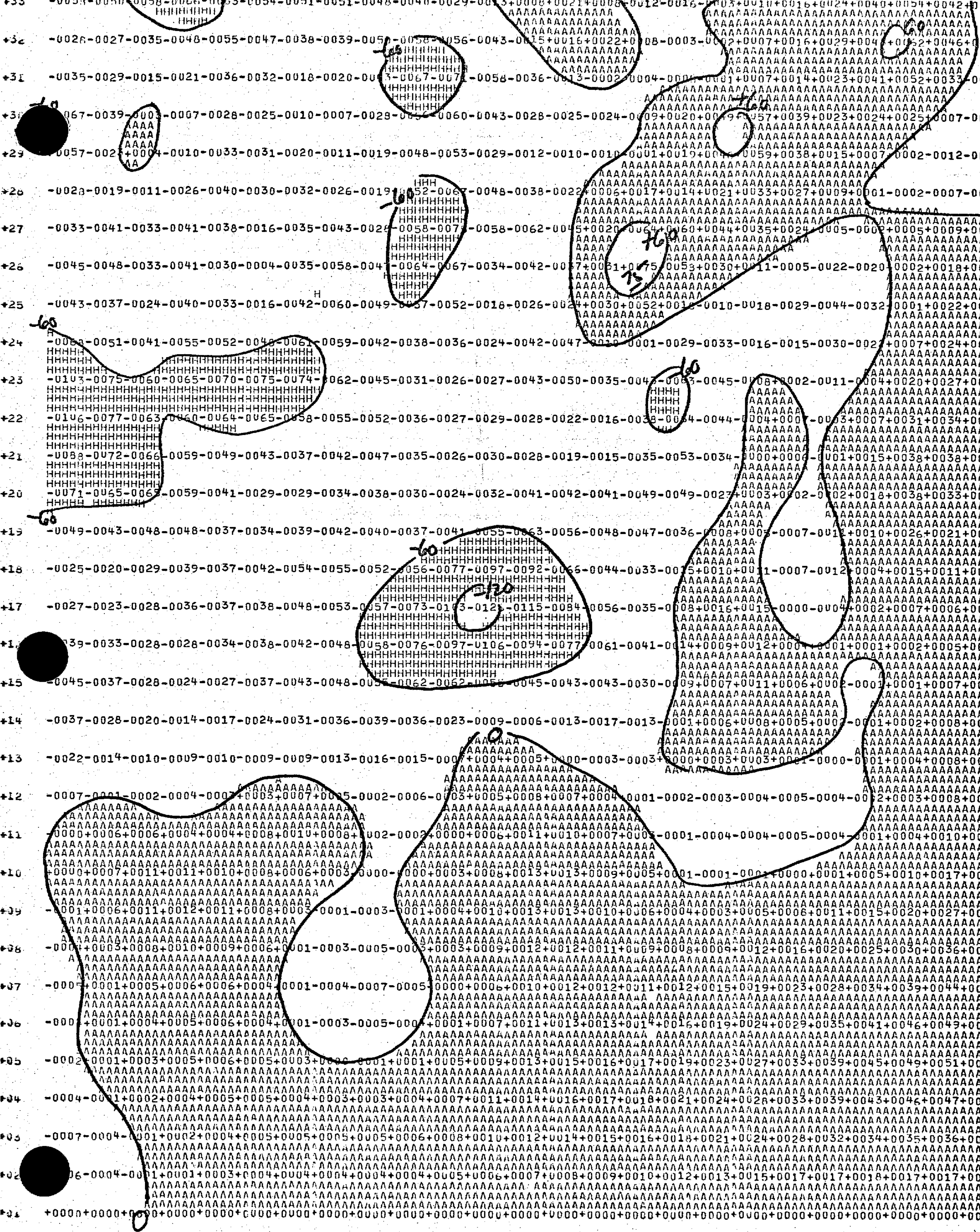


FIG 5