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Let's Try Non-linear Initialization Simply and Correctly (at Least to Start With)

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Abstract

The Machenhauer non-linear initialization method as implemented at NMC is shown to contain theoretical inconsistencies that probably explain some of the convergence problems and restrictions that NMC has encountered. The general remedy is to follow the Baer approach by (a) first setting all fast mode amplitudes to zero and then (b) doing only one Machenhauer iteration. Two exceptions to this prescription occur with intense orographic uplift and with intense release of latent heat. In these cases, the large vertical velocities found in gravity waves will be needed to get the correct starting field in step (a).

1. Background.

Machenhauer (1976) suggested a way to eliminate "noise" from unwanted high frequency oscillations in a numerical prediction. This was accomplished by first considering the initial data as mapped onto the spatial modes of a reference atmosphere. Each such mode has its own frequency of oscillation when it is acted on by the linearized equations of motion. With some discretion, these modes can be divided into

S. Slow modes (small frequency)

F. Fast modes (large frequency)

Roughly speaking, S consists of Rossby waves and F of gravity waves.

One can write the full prediction equations symbolically, in the following form:

$$\frac{dS}{dt} = -i\Omega_S S + i P_S \cdot NL(F,S), \qquad (1.1a)$$

$$\frac{dF}{dt} = -i\Omega_F F + i P_F \cdot NL(F,S) \qquad (1.2)$$

S and F represent the amplitude of a slow and a fast mode, with frequencies Ω_S and Ω_F, respectively. NL represents the spatial field of all non-linear tendencies. P_S and P_F denote a projection of the latter field onto the spatial structure of mode S and mode F.

Previous initialization methods using normal modes of a reference atmosphere had simply set $F=0$. Machenhauer's extremely valuable insight was to recognize that a better initial state might be obtained through choosing F at $t=0$ by ignoring the left side of (1.2):

$$F(t=0) = \frac{1}{\Omega_F} P_F \cdot NL(F,S) \qquad (1.3)$$

where F and S in NL(F,S) are obtained from the input analysis. This single step turned out to be not quite satisfactory in Machenhauer's empirical tests. To improve the results, Machenhauer found it useful in his barotropic model to iterate this process:

$$F_{m+1}(t=0) = \frac{1}{\Omega_F} P_F \cdot NL(F_m, S) \tag{1.4}$$

Use of this method at NMC has been attended by considerable problems. Most striking is the fact that (1.4) sometimes will not converge unless it is restricted to the external and first internal vertical modes. Why should this limitation exist, when the old-fashioned quasi-geostrophic method (Phillips, 1962, p. 157) gives a unique non-iterative answer for any detail in the vertical?

In this Note, I will illustrate two points that are implicit in the published literature, but seem to merit stronger emphasis:

I. F in the NL term on the right side of (1.3) must be zero.

II. The iteration procedure (1.4) is not justified beyond (1.3) unless accompanied by consideration of the left side of (1.2).

These requirements will be demonstrated by considering an extremely simple model. This model will be analyzed in three ways:

a. Rossby number expansion of field variables.

b. Complete exact solution of the system.

c. Rossby number expansion of the normal mode equations.

The method is in the spirit of the approach by Baer (1977), but should be much easier to understand. The results I and II are also compatible with the conditions which Leith (1980) had to assume in order to demonstrate that this form of the Machenhauer method was equivalent to the conventional quasi-geostrophic system.

Several exceptions that occur when $NL(F, S)$ is large are discussed briefly in a closing section.

2. Barotropic Free Surface Model on a Constant f Plane.

The equations for this model will be linear, with a uniform basic current, $\bar{u}$:

$$\frac{\partial u'}{\partial t} - f v' + \frac{\partial \phi'}{\partial x} = - \bar{u} \frac{\partial u'}{\partial x}, \qquad (2.1a)$$

$$\frac{\partial v'}{\partial t} + f u' = - \bar{u} \frac{\partial v'}{\partial x}, \qquad (2.2b)$$

$$\frac{\partial \phi'}{\partial t} + c^2 \frac{\partial u'}{\partial x} = - \bar{u} \frac{\partial \phi'}{\partial x}. \qquad (2.1c)$$

The ground has been assumed, for simplicity, to slope downward in the y-direction at the rate $d\bar{z}/dy = -f\bar{u}/g$, so that (2.1c) does not contain a term $v' \, dc^2/dy$. An alternate interpretation of (2.1) in terms of a modal representation of a three-dimensional atmospheric model, would be that it models the interaction between

(a) $\bar{u}$, representing the lowest order vertical mode of longitudinal wave number zero, with

(b) the variables, (u', v', ϕ') corresponding to an arbitrary vertical mode m and a single zonal wave number k. These are specified by the choice of c^2 and k:

$$c^2 = c_m^2,$$
$$u', v', \phi' \propto \exp(ikx). \qquad (2.2)$$

$m=0$, with $c_0 \sim 320\ m\ sec^{-1}$, would be an external mode. The effect on $\bar{u}$ of the wave mode is obviously ignored, but is irrelevant for this illustration. The terms on the right sides of (2.1) will represent the NL field in equations (1.1) and (1.2). Although simple, they imitate the most important and pervasive non-linearity in the behavior of the atmosphere.

To simplify the notation and display the different approximation procedures to define initial data, we make the variables non-dimensional in the customary manner of the quasi-geostrophic expansion (Monin, 1958; Charney, 1962; Phillips, 1962)

$$\tau = (k\bar{u})t \qquad (2.3a)$$

$$\xi = kx \qquad (2.3b)$$

$$u = u'/\bar{u} \qquad (2.3c)$$

$$v = v'/\bar{u} \qquad (2.3d)$$

$$\phi = \phi'(k/f\bar{u}) \qquad (2.3e)$$

$$Ro = k\bar{u}/f \qquad (2.3f)$$

$$\sigma = k^2c^2/f^2 \qquad (2.3g)$$

The non-dimensional equivalents of (2.1) are

$$Ro\frac{\partial u}{\partial \tau} - v + \frac{\partial \phi}{\partial \xi} = -Ro\frac{\partial u}{\partial \xi}, \qquad (2.4a)$$

$$Ro \frac{\partial v}{\partial t} + u \qquad = -Ro \frac{\partial v}{\partial \xi}, \tag{2.4b}$$

$$Ro \frac{\partial \phi}{\partial t} + \sigma \frac{\partial u}{\partial \xi} \qquad = - Ro \frac{\partial \phi}{\partial \xi} . \tag{2.4c}$$

Ro is the Rossby number, while σ measures the squared ratio of the length scale k^{-1} to the Rossby radius of deformation c/f.

We shall assume that all these variables have a time behavior characterized by the "slow" advective time t, i.e. that $\partial/\partial t$ is of order unity. We also assume that $\partial/\partial \xi$ is of order unity, that Ro is small and that σ is of order zero with respect to Ro. To simplify the ensuing mathematics we then introduce the Fourier representation:

$$\begin{aligned} u &= Re\left[U(t) e^{i\xi} \right], \\ v &= Re\left[i\, V(t) e^{i\xi} \right], \\ \phi &= Re\left[\sqrt{\sigma}\, H(t) e^{i\xi} \right]. \end{aligned} \tag{2.5}$$

Equations (2.4) can now be written as

$$Ro\, \dot{V} - iU \qquad = -i\, Ro\, V, \tag{2.6a}$$

$$Ro\, \dot{H} + i\sqrt{\sigma}\, U \qquad = -i\, Ro\, H, \tag{2.6b}$$

$$Ro\, \dot{U} - iV + i\sqrt{\sigma}\, H \qquad = -i\, Ro\, U, \tag{2.6c}$$

where $\dot{( )}$ denotes $d( )/dt$. The terms on the rhs of these equations are the "non-linear" interactions.

We first perform a conventional Ro expansion of (2.6) for the field variables, V, H, U. Equations (2.6a) and (2.6b) both show that U is at most of order Ro. If we set $U \propto Ro$, the parameter Ro would cancel in (2.6a) and (2.6b). We will then have to deal only with

$$\epsilon = Ro^2 = k^2 \bar{u}^2 / f^2, \tag{2.7}$$

and only in (2.6c).

We carry out the quasi-geostrophic expansion by setting

$$[u(\tau), V(\tau), H(\tau)] = \sum_{n=0}^{\infty} [Ro\, U_n, V_n, H_n]\, \epsilon^n, \tag{2.8}$$

where U_n, etc. are functions of τ. This yields the following approximation sequence upon substitution in (2.6):

$$\dot{V}_n = i U_n - i V_n, \tag{2.9a}$$

$$\dot{H}_n = -i\sqrt{\sigma}\, U_n - i H_n, \tag{2.9b}$$

$$\dot{U}_{n-1} = i V_n - i\sqrt{\sigma}\, H_n - i U_{n-1}. \tag{2.9c}$$

These are valid at all n for all τ, according to our assumptions.

At $n=0$, (2.9c) gives the geostrophic relation

$$V_0 = \sqrt{\sigma}\, H_0 \tag{2.10a}$$

(2.9a) and (2.9b) then give

$$i U_0 = i V_0 + \dot{V}_0 \tag{2.10b}$$

$$i \sigma U_0 = -\sqrt{\sigma}(i H_0 + \dot{H}_0) = -(i V_0 + \dot{V}_0) \tag{2.10c}$$

Together, these demonstrate that

$$U_0 = 0 , \tag{2.11a}$$

$$(V_0, H_0) = (V_0(0), H_0(0))\, e^{-it} . \tag{2.11b}$$

To go to higher order it is efficient to first combine (2.9a) and (2.9b)[1]:

$$(1+\sigma) U_n = (V_n - \sqrt{\sigma} H_n) - i(\dot{V}_n - \sqrt{\sigma} \dot{H}_n), \tag{2.12a}$$

and rewrite (2.9c) as

$$(V_n - \sqrt{\sigma} H_n) = U_{n-1} - i \dot{U}_{n-1} . \tag{2.12b}$$

It then becomes immediately apparent by induction that the relations (2.11a) and (2.11) hold for <u>all</u> orders of n. We therefore find from (2.8) that the initial conditions for only "slow" time behavior in (2.4) or (2.6) requires

$$U(t=0) = 0,$$
$$V(t=0) = \sqrt{\sigma}\, H(t=0). \tag{2.13}$$

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[1]U is proportional to the divergence. (2.12a) is equivalent to the quasi-geostrophic "omega equation" for this model, and (2.12b) to the "balance equation".

Let us now consider the complete behavior for arbitrary initial values of U, V, H. This will tell us if (2.13) is correct. To do this we must solve (2.6), in which the slow time assumption has not yet been enforced. We define the column matrix

$$\underset{\sim}{X} = (V, H, U) \tag{2.14a}$$

so that (2.6) can be written as

$$Ro\,\dot{\underset{\sim}{X}} = -i\,B\,\underset{\sim}{X}\,, \tag{2.14b}$$

where B is the symmetric matrix

$$B = \begin{pmatrix} Ro & 0 & -1 \\ 0 & Ro & \sqrt{\sigma} \\ -1 & \sqrt{\sigma} & Ro \end{pmatrix}. \tag{2.14c}$$

(The diagonal Ro terms come from the "non-linear" advection terms.)
The three eigenvalues of B are

$$\lambda_1 = Ro\,, \tag{2.15a}$$

$$\lambda_2 = Ro + \sqrt{1+\sigma} = Ro + \lambda_0\,, \tag{2.15b}$$

$$\lambda_3 = Ro - \sqrt{1+\sigma} = Ro - \lambda_0\,. \tag{2.15c}$$

The frequencies in τ units of system (2.14b) are given by λ_j/Ro.
The parameter $\lambda_0 = \sqrt{1+\sigma}$ is recognizable as the frequency of an internal gravity wave in the time units defined by τ in (2.3a):

$$\frac{\lambda_0 t}{Ro} = \frac{\sqrt{f^2+k^2c^2}}{k\bar{u}}\, t = \sqrt{f^2+k^2c^2}\; t \,. \tag{2.16}$$

The three normalized orthogonal eigenvectors have the following components for the variables (V, H, U):

$$\underset{\sim}{w}_1 = (\sqrt{\sigma},\, 1,\, 0)/\lambda_0 \,, \tag{2.17a}$$

$$\underset{\sim}{w}_2 = (-1,\, \sqrt{\sigma},\, \lambda_0)/\sqrt{2}\,\lambda_0 \,, \tag{2.17b}$$

$$\underset{\sim}{w}_3 = (-1,\, \sqrt{\sigma},\, -\lambda_0)/\sqrt{2}\,\lambda_0 \,. \tag{2.17c}$$

(Note that these do not depend on Ro, and are identical with the eigen-. vectors for B if $Ro = 0$.)

The slow solution is that for λ_1 and $\underset{\sim}{w}_1$:

$$V = K\sqrt{\sigma}\, e^{-it} \,, \tag{2.18a}$$

$$H = K\, e^{-it} \,, \tag{2.18b}$$

$$U = 0 \,. \tag{2.18c}$$

The quasi-geostrophic expansion of the field variables that resulted in (2.13) as the initial conditions for U, V, and H has therefore determined precisely those initial conditions for which the complete equation (2.6) will produce only the slow solution (2.18).

For comparison, we now examine the initialization problem solved in (2.6)-(2.13) by using the normal mode approach. We express $\underset{\sim}{X}(t)$ as

$$\begin{aligned} \underset{\sim}{X}(t) &= (V(t), H(t), U(t)) \\ &= \alpha(t)\underset{\sim}{w}_1 + \beta(t)\underset{\sim}{w}_2 + \gamma(t)\underset{\sim}{w}_3 . \end{aligned} \tag{2.19}$$

The eigenvectors $\underset{\sim}{w}_j$ are defined by (2.17). Here we emphasize that they are also the eigenvectors of the resting reference state matrix:

$$B(Ro=0) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & \sqrt{\sigma} \\ -1 & \sqrt{\sigma} & 0 \end{pmatrix} \tag{2.20}$$

It is in this sense that they are used in (2.19). Upon applying (2.19) to (2.6), the following equations result

$$\dot{\alpha} = \qquad\qquad -i\alpha \tag{2.21a}$$

$$\dot{\beta} = -\frac{i\lambda_0}{Ro}\beta \qquad -i\beta \tag{2.21b}$$

$$\dot{\gamma} = +\frac{i\lambda_0}{Ro}\gamma \qquad -i\gamma \tag{2.21c}$$

In more complicated systems, the $-i\alpha, -i\beta, -i\gamma$ "non-linear" terms would couple these equations with one another.

We apply an expansion procedure similar to (2.8)-(2.12) with the same requirement that $(\dot{\ })$ is of order 1. Our model system is especially simple since each of (2.21) refers to only one model amplitude. The interesting ones are (2.21b) and (2.21c). We define the small parameter

$$\delta = Ro/\lambda_0 , \tag{2.22a}$$

and again set up a power series needed to consider, for example, (2.21b):

$$\beta(t) = \sum_{n=0}^{\infty} \beta_n(t)\,\delta^n . \tag{2.22b}$$

(2.21b) becomes

$$\delta\dot{\beta} = -i\beta - i\delta\beta . \tag{2.22c}$$

Insertion of (2.22b) gives, for $n = 0$, ..., at all t,

$$\dot{\beta}_{n-1} = -i\beta_n - i\beta_{n-1} . \tag{2.23}$$

Since n starts with 0, this immediately yields the conclusion

$$\beta_0(t) = 0,$$

and then, by induction,

$$\beta_n(t) = 0.$$

A similar conclusion follows for $\gamma_n(t)$. From this we conclude that both β and γ must vanish in order to give only a slow time solution. At $t=0$, in particular, this means that the initial value of V, H, and U are described by only the $\underset{\sim}{w}_1$ eigenvector of (2.17a).

Thus we again get, correctly,

$$V(0) = \sqrt{\sigma}\, H(0), \quad U(0) = 0, \tag{2.24}$$

as in (2.13).

Let us now consider the Machenhauer approach to the modal system (2.21b). In that approach $\dot{\beta}$ on the left side is completely ignored and the second term $(-i\beta)$ on the rhs represents the projection onto vector $\underset{\sim}{w}_2$ of the complete set of forecast time derivatives. In our case his method gives rise to an iteration procedure (see equation (1.4):

$$i \frac{\lambda_0}{R_0} \beta_{l+1} = \underset{\sim}{w}_2 \cdot NL(\alpha_0, \beta_l, \gamma_l) = -i \beta_l \tag{2.25}$$

After a suitable number of iterations, β_l replaces the initial value of β in the modal representation of the input data.

Equation (2.25) has the solution

$$\beta_l = \left(\frac{-k\bar{u}}{\sqrt{f^2+k^2c^2}} \right)^l \beta_0 \tag{2.26}$$

where β_0 is the projection on $\underset{\sim}{w}_2$ of the initially given $\underset{\sim}{X}(t=0)$.

Having gone through the geostrophic expansion procedure (2.6)-(2.13), the complete solution procedure (2.14)-(2.19), and the modal method (2.22)-(2.24), we know that the correct solution to get only the slow mode solution is that β should be zero. The Machenhauer techniques in this simple case approaches the correct answer only if either the advective frequency is small compared to the gravity wave frequency

$$|k\bar{u}| < \sqrt{f^2+k^2c^2}, \tag{2.27}$$

or, if the input data already has no projection on $\underset{\sim}{w}_2$ or $\underset{\sim}{w}_3$:

$$\beta_{l=0} = 0, \gamma_{l=0} = 0. \tag{2.28}$$

The version of the Machenhauer procedure in use at NMC does not use (2.28). Therefore we can conclude that:

<u>Much of the work done in the NMC Machenhauer iteration scheme represents a laborious attempt to correct for the failure to enforce (2.28) at the start of the process.</u>

In doing so it will fail to converge when (2.27) is violated. This can easily happen for the high order internal modes that have small values of λ_0 in the spherical three-dimensional set of normal modes.

A condition similar to (2.27) has also been encountered by Gollvik (1980) in his study of initialization on an equatorial β-plane. His reported criterion (p. 21) of

$$\varepsilon = \frac{U}{2\Omega a} = .04$$

$$\kappa = \frac{4\Omega^2 a^2}{c^2} < 2750 = \kappa_c$$

corresponds to convergence only if

$$U^2 < (\kappa_c \varepsilon^2) c^2 = 4.4\, c^2$$

A second conclusion can be reached from (2.22c):

$$\delta \dot{\beta} = -i\beta \qquad - i \delta \underset{(NL)}{\beta}\, . \tag{2.22c}$$

The left side is completely ignored in the Machenhauer iteration scheme. In our simple model we have seen that the correct solution was simply $\beta = 0$. Generalizing our view of this equation to a more general system requires only that the NL term $-i\delta\beta$ include not just β, but contributions from many other components. This more general form will be

$$\delta\dot{\beta} = -i\beta + i\delta\, F(\alpha, \gamma, \beta, \text{etc}). \tag{2.29}$$

F will be of order zero. (But see Section 3.) In terms of an expansion in δ,

$$\beta(t) = \sum_{m=0} \beta_m(t)\,\delta^m, \tag{2.30}$$

we obtain

$$\dot{\beta}_{m-1} = -i\,\beta_m + i\,F_{m-1}, \tag{2.31}$$

where F_{m-1} uses β_{m-1}.

At $m=0$ we again get[1]

$$\beta_0 = 0. \tag{2.32}$$

At $m=1$ we obtain

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[1]Daley (1978) removed the initial divergence in his tests. This amounts only to making the difference $\beta - \gamma$ equal to zero in (2.21)-(2.26). It does not remove the sum $\beta + \gamma$ which represents the difference between vorticity and the Laplacian of the geopotential field.

$$\beta_1 = F_0 + i\dot{\beta}_0 = F_0 . \tag{2.33}$$

This would be the first Machenhauer iteration if one started it with $\beta = 0$. But, at $m = 2$, we get

$$\beta_2 = F_1 + i\dot{\beta}_1 = F_1 + i\dot{F}_0 \tag{2.34}$$

The time derivative cannot generally be ignored after the first (corrected) Machenhauer iteration. Our second conclusion is therefore

Even if the Machenhauer process is corrected by starting with zero gravity mode amplitude, only one iteration is meaningful.

Further improvement must consider the $\dot{\beta}$ term, and we would enter upon the Baer approach.

3. Concluding Remarks.

The purpose of this note is primarily to point out how the Machenhauer method, as it has been implemented at NMC, must be changed. This correction is only a first step. Some of the problems remaining in initialization are as follows.

a. The correct choice of the $m = 0$ field -- i.e. the fixed values of the slow mode amplitudes -- is important. A variational analysis is desirable in this step so that proper attention is paid to the relative accuracy of temperature, surface pressure, and wind data in the input analysis, and to their accuracies in different geographic regions.

Daley (1978) has explored such an approach. However, his approach assumes that the analysis weights are known very accurately[1]. The uncertainties in them seem to me to justify combining the variational procedure only with the first step in the initialization process, that of obtaining the (fixed) amplitudes of the slow modes.

b. The effect of non-transient heating fields must be considered. This is especially critical for the release of latent heat in low latitudes, where this heating must be balanced by large upward vertical motions if a slow time solution is to result. This creates a dilemna, since only the fast modes have appreciable vertical velocity, and they have been zeroed out in step (2.32). In these circumstances, moreover, F in (2.29) will no longer be of order one. A special treatment is evidently needed to get the fast modes needed to balance these intense heating fields. This task will be simplified to the extent that only those "fast" modes having maximum amplitude near the equator need be considered.

c. A similar problem will occur if orographic uplift creates vertical velocities so large that the tendencies in NL can only be matched by gravity waves (see Phillips, 1962, p. 138). These regions are scattered over the globe and their existence will vary from day to day depending on the strength of the horizontal velocity perpendicular to the orographic slope. Their interpretation is also confounded by the presence of large vertical truncation error near the tropopause in the computation of the horizontal pressure force in sigma coordinates. A satisfactory treatment of these regions in initialization will therefore be of necessity more empirical than that needed to resolve problem b.

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[1]He does a variational technique in <u>each</u> step of a Machenhauer (preferably, Baer) iteration.

d. If the present NMC system is replaced by one or several Baer steps, we may still expect some "convergence" problems even though (2.27) is no longer relevant. For example, Daley (1978, p. 210; see also Tribbia, 1979) encountered difficulties in a barotropic model in regions when the ellipticity condition for the balance equation is violated. This is a condition on smallness of the Rossby number in anticyclonic vorticity, and is relevant even in a barotropic model. Another condition in synoptic scale baroclinic motions will involve the Richardson number:

$$Ri = g \frac{d \ln \theta}{dz} \div \left(\frac{\partial u}{\partial z} \right)^2 \gtrsim O(1) \quad (3.1)$$

Although $Ri \sim 40$ for normal synoptic motions, it can become 0(1) in sharp frontal zones. It can be even smaller when saturation exists in air of nearly uniform equivalent potential temperature. These are the situations which can give rise to extremely rapid baroclinic cyclogenesis (and on smaller than normal length scales); we cannot expect routine application of non-linear initialization to be useful in these regions.

e. In initialization of the initial field for a real forecast, the goal is, to a considerable extent, cosmetic. This is because the variational analysis (a, above) will affect weather forecasts, while (2.33)-(2.34) mostly eliminate distracting noise by anticipating the dispersive adjustments that would otherwise occur in the early part of the forecast. But for the data assimilation system (the NMC "final cycle"), the tropical and subtropical moisture, mass, and momentum fields are, because of lack of data, mainly determined by the "general circulation" behavior of the forecast model used in the assimilation cycle. For example, the mean meridional circulation in the atmosphere and in the assimilation model moves moisture into the latitudes of the intertropical convergence zone. This circulation is forced by the fields of heating, eddy transports, and friction. The first two of these will be accounted for in the initialization process, if a satisfactory solution of point b is achieved. Friction, while equally important to the heating overall, operates on larger horizontal space scales than does the tropical convective heating. As such it probably does not destroy the condition $F \sim 1$ in (2.29). Instead the problem here will be connected with the fact that the normal modes being used are for a friction-less atmosphere, and tend to have a maximum of horizontal velocity at the ground. These would be completely inconsistent with a no-slip frictional boundary condition at the ground. It remains to be seen how well they can represent the usual meteorological drag law, and Ekman-like ageostrophic circulations, in low latitudes.

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