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# Linear Computational Stability of the Core-Contained 4th Order LFM

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## 1. INTRODUCTION

During early March 1983 evidence surfaced that suggested the presence of linear computational stability within the LFM model. The numerical method used to integrate the model equations has evolved over the years since the fundamental analysis of Brown and Campana (1978) appeared. In particular the model now uses fourth-order accurate, finite-difference approximations and a rather large diffusion coefficient, especially near the boundaries. In an effort to shed some light on the observed behavior of the model, I have carried out a linear, computational stability analysis taking into account the presently used methods.

## 2. THE BASIC EQUATIONS

The momentum and continuity equations form the basic set for this analysis. We use the linearized, one-dimensional versions, neglecting rotation.

$$\frac{\partial u}{\partial t} = -U\frac{\partial u}{\partial x} - g\frac{\partial h}{\partial x} + K\frac{\partial^2 u}{\partial x^2} \tag{1.a}$$

$$\frac{\partial h}{\partial t} = -U\frac{\partial h}{\partial x} - H\frac{\partial u}{\partial x} + K\frac{\partial^2 h}{\partial x^2} \tag{1.b}$$

If we define

$$\begin{aligned} p &\equiv h/H \\ v &\equiv u/c \\ c &\equiv \sqrt{gH} \end{aligned} \tag{2. }$$

the equations take the form.

$$\frac{\partial v}{\partial t} = -U\frac{\partial v}{\partial x} - c\frac{\partial p}{\partial x} + K\frac{\partial^2 v}{\partial x^2} \tag{3.a}$$

$$\frac{\partial p}{\partial t} = -U\frac{\partial p}{\partial x} - c\frac{\partial v}{\partial x} + K\frac{\partial^2 p}{\partial x^2} \tag{3.b}$$

The ~~in~~dependent variables v and p are assumed to depend upon x via the functional form $e^{ikx}$ with k the wave number. Using this assumption in eqs. 3,

$$\frac{\partial v}{\partial t} = -ikUv - ikcp - Kk^2 v \qquad (4.a)$$

$$\frac{\partial p}{\partial t} = -ikUp - ikcv - Kk^2 p \qquad (4.b)$$

## 2.1 SPACE TRUNCATION

To account for the actual use of finite difference approximations for the derivatives, the analytical value of k should be replaced as follows:

For 2nd order, first derivative,

$$k = \sin(k\Delta x)/\Delta x \qquad (5.a)$$

For 4th order, first derivative,

$$k = [87 \sin(k\Delta x) - 12 \sin(2k\Delta x) + .33 \sin(3k\Delta x)] \div [64\Delta x] \qquad (5.b)$$

For 2nd order, second derivative,

$$k = 2(1-\cos(k\Delta x))/(\Delta x)^2 \qquad (5.c)$$

## 2.2 PRESSURE GRADIENT AVERAGING

The time-averaging of the pressure gradient in the momentum equation may be used in conjunction with the 'leap-frog' approximation for time integration. One simply solves the continuity equation first. The equations take the form:

$$p^{n+1} - p^{n-1} = -2\Delta t\left[ikUp^n + ikcv^n + Kk^2 p^{n-1}\right] \qquad (6.a)$$

$$v^{n+1} - v^{n-1} = -2\Delta t\left[ikUv^n + ikc(\alpha p^{n+1} + \alpha p^{n-1} + \hat{\alpha} p^n) + Kk^2 p^{n-1}\right] \qquad (6.b)$$

The coefficient $\hat{\alpha} = 1-2\alpha$. It is known that with U=K=0 and $0<\alpha<1/4$ the allowable time step for use with the system 6 is greater than that allowed by the usual leap-frog scheme. In the case $\alpha=1/4$, one finds that the allowable time step is a factor of two greater than in the usual leap-frog scheme. There is however a linear, not exponential, growth permitted for the temporal, computational mode. Since past experience indicated that such a mode can lead to non-linear instability, the system of equations was augmented by use of the Robert-Asselin time filter.

## 2.3 TIME FILTERED, PRESSURE GRADIENT AVERAGING

Except for our inclusion of the diffusion term and the advection term the equations to be analyzed below are identical to those studied by Brown and Campana. The system of equations is

$$\hat{p}^{n+1} = p^{n-1} - 2\Delta t\left[ikU\hat{p}^{n} + ikc\hat{v}^{n} + Kk^2 p^{n-1}\right] \tag{7a}$$

$$\hat{v}^{n+1} = v^{n-1} - 2\Delta t\left[ikU\hat{v}^{n} + ikc\left(\alpha\hat{p}^{n+1} + \alpha p^{n-1} + \hat{\alpha}\hat{p}^{n}\right) + Kk^2 p^{n-1}\right] \tag{7b}$$

$$p^{n} = \hat{\beta}\hat{p}^{n} + \beta\hat{p}^{n+1} + \beta p^{n-1} \tag{7c}$$

$$v^{n} = \hat{\beta}\hat{v}^{n} + \beta\hat{v}^{n+1} + \beta v^{n-1} \tag{7d}$$

The coefficient $\hat{\beta} = 1 - 2\beta$ . The value of $\beta$ used in the LFM is 0.075. Brown and Campana showed that a near optimal choice of $\alpha$ follows from the formula

$$\alpha = 0.25 * (\beta^2 + 1) * (\beta + 1) \tag{8}$$

Before proceeding to the analysis of the system of equations 7 it is convenient to define $\mu = kU\Delta t$ , $\nu = kc\Delta t$, $\sigma = k^2 K\Delta t$

$$\hat{\sigma} = 1 - 2\sigma \tag{9}$$

## 3. ALGEBRAIC DERIVATION OF STABILITY CRITERION

The equations 7 become with 9,

$$\hat{p}^{n+1} = \hat{\sigma} p^{n-1} - 2\mu\hat{p}^{n} - 2\nu\hat{v}^{n} \tag{10.1}$$

$$\hat{v}^{n+1} = \hat{\sigma} v^{n-1} - 2\mu\hat{v}^{n} - 2\nu\left[\alpha\hat{p}^{n+1} + \alpha p^{n-1} + \hat{\alpha}\hat{p}^{n}\right] \tag{10.2}$$

$$p^{n} = \hat{\beta}\hat{p}^{n} + \beta\hat{p}^{n+1} + \beta p^{n-1} \tag{10.3}$$

$$v^{n} = \hat{\beta}\hat{v}^{n} + \beta\hat{v}^{n+1} + \beta v^{n-1} \tag{10.4}$$

but for convenience in the algebraic elimination of non-time averaged quantities (those with carats), the equations are written again, raising n to n+1:

$$\hat{p}^{n+2} = \hat{\sigma} p^{n} - 2\mu \hat{p}^{n+1} - 2\nu \hat{v}^{n+1} \tag{10.5}$$

$$\hat{v}^{n+2} = \hat{\sigma} v^{n} - 2\mu \hat{v}^{n+1} - 2\nu\left[\alpha \hat{p}^{n+2} + \alpha p^{n} + \hat{\alpha}\hat{p}^{n+1}\right] \tag{10.6}$$

$$p^{n+1} = \hat{\beta}\hat{p}^{n+1} + \beta \hat{p}^{n+2} + \beta p^{n} \tag{10.7}$$

$$v^{n+1} = \hat{\beta}\hat{v}^{n+1} + \beta \hat{v}^{n+2} + \beta v^{n} \tag{10.8}$$

Now we use eqs. 10.2 and 10.6 in eq. 10.8 to replace $\hat{v}^{n+1}$ and $\hat{v}^{n+2}$. After collecting terms, one may then use eqs. 10.3, 10.4 and 10.7 to eliminate all reference to non-time averaged variables. A final collection of terms yields

$$v^{n+1} - [\beta(1+\hat{\sigma}) - 2i\mu] v^{n} - [\hat{\beta}\hat{\sigma} + 2i\beta\mu] v^{n-1} + 2i\nu\left[\alpha p^{n+1} + \hat{\alpha} p^{n} + (\alpha-\beta) p^{n-1}\right] = 0 \tag{11}$$

Following an analogous method, we use eqs. 10.1 and 10.5 in eq. 10.7 to replace $\hat{p}^{n+1}$ and $\hat{p}^{n+2}$. After collecting terms, equations 10.3 and 10.4 permit one to eliminate all reference to non-time averaged variables. The final equation may be expressed as

$$p^{n+1} - [\beta(1+\hat{\sigma}) - 2i\mu] p^{n} - [\hat{\beta}\hat{\sigma} + 2i\beta\mu] p^{n-1} + 2i\nu\left[v^{n} - \beta v^{n-1}\right] = 0. \tag{12}$$

One now takes the solution in the form

$$p^{n} = P\lambda^{n} \qquad v^{n} = V\lambda^{n} \tag{13}$$

and obtains the determinant of the coefficients of P and V (appearing in eqs. 11 and 12 after 13 has been introduced). The determinant must vanish and therefore one has the characteristic polynomial in $\lambda$ :

$\lambda$

$$\lambda^4 + c_1\lambda^3 + c_2\lambda^2 + c_3\lambda + c_4 = 0 . \qquad (14)$$

The coefficients c are

$$c_1 = 4\nu^2\alpha - 2\beta(1+\hat{\sigma}) + 4i\mu$$

$$c_2 = 4\nu^2(\hat{\alpha} - \alpha\beta) - 2\hat{\sigma}\hat{\beta} + \beta^2(1+\hat{\sigma})^2 - 4\mu^2 - i[4\mu\beta(2+\hat{\sigma})]$$

$$c_3 = 4\nu^2(\alpha - \beta - \hat{\alpha}\beta) + 2\hat{\sigma}\hat{\beta}\beta(1+\hat{\sigma}) + 8\beta\mu^2 + i[4\mu\{\beta^2(1+\hat{\sigma}) - \hat{\sigma}\hat{\beta}\}]$$

$$c_4 = 4\nu^2\beta(\beta - \alpha) + \hat{\sigma}^2\hat{\beta}^2 - 4\beta^2\mu^2 + i\,4\mu\beta\hat{\beta}\hat{\sigma} .$$

In order for the numerical integration to be linearly stable one must have

$$|\lambda| < 1$$

We have compared the coefficients c with those given by Brown and Campana by setting $\hat{\sigma} = 1$ $(\sigma = 0)$, $\mu = 0$ . Using the notation in this note, the coefficients in eq. 17 of Brown and Campana are:

$$c_1 = 4\nu^2\alpha - 4\beta$$

$$c_2 = 4\nu^2(\hat{\alpha} - \alpha\beta) - 2\hat{\beta} + 4\beta^2$$

$$c_3 = 4\nu^2(\alpha - 2\beta + 2\alpha\beta) + 4\beta\hat{\beta}$$

$$c_4 = \hat{\beta}^2 + 4\nu^2\beta(\beta - \alpha)$$

The only apparent discrepancy is in the coefficient of $4\nu^2$ in the equation for $c_3$.

For the relations to be identical one must have

$$\alpha - 2\beta + 2\alpha\beta \equiv \alpha - \beta - \hat{\alpha}\beta$$

Since $\hat{\alpha} = 1 - 2\alpha$ the right hand side may be expanded

$$= \alpha - \beta - \beta(1 - 2\alpha)$$

$$= \alpha - 2\beta + 2\alpha\beta$$

This shows that there is no discrepancy.

When we compare our equation with that in the paper by Schoenstadt and Williams (1976) we find that their equation (36) which should reduce to Brown and Campana's eq. (17) (upon setting their S to B&C's D and $\gamma$ to b) does not do so. Using the notation of this note the coefficients of Schoenstadt and Williams are,

$$c_1 = 4\nu^2\alpha - 4\beta$$

$$c_2 = 4\nu^2(1-2\alpha-2\alpha\beta) + 4\beta + 4\beta^2 - 2$$

$$c_3 = 4\nu^2[\alpha + 2\alpha\beta - 2\beta] + 4\beta\hat{\beta}$$

$$c_4 = 4\nu^2\beta(\beta-\alpha) + \hat{\beta}^2$$

In the coefficient of $4\nu^2$ in the formula for $c_2$, there is an additional $-\alpha\beta$ present. The remaining terms all agree with our equations and that of Brown and Campana; we therefore conclude that our equation is correct. The additional terms involving $\mu$ in Schoenstadt and Williams equation (39) do not agree with our equations either. The reason for these discrepancies is not certain, but it may be due to Schoenstadt and Williams' use (cf their eq. 35) of an analytic, rather than simple algebraic, approach to determining the role of the Robert-Asselin time filter. We have checked our algebra and believe it to be correct.

## 4. RESULTS

We have solved the characteristic equation for a variety of conditions. The results are tabulated in Table 1 below. In all cases, the pressure gradient averaging weight $\alpha$ and the Robert-Asselin time averaging weight $\beta$ were computed as in the operational model. The values are $\beta = 0.075$ and $\alpha = 0.27$.

Most of the computations used the parameters appropriate for the low latitude boundary zone of the model. There the grid size $\Delta x$ is a minimum near 120 km, and the diffusion coefficient is a maximum, $K \sim 1.8 \times 10^6\ m^2 sec^{-1}$.

The operational model uses a 400 sec time step and approximates both gravity wave and advection terms using 4th order accurate finite-difference approximations. The diffusion term uses a 2nd order accurate approximation to the Laplacian (5 point) operator. We are not certain of the appropriate value for the gravity wave phase velocity c, but we assume that our use of 330 m $sec^{-1}$ errs on the side of conservatism.

As shown by entry #1 in the Table, the operational values gave us a unstable solution. Entry #2 in the table shows that when diffusion is removed the operational system is stable as long as the wind doesn't exceed 10 m/sec. Entry #3 repeats the computation without diffusion, but introduces second-order accurate finite differences. In this case the solution is stable for all wind speeds tested (up to 70 m/sec).

It appears therefore that two effects: diffusion and the numerical accuracy of the approximations are both important.

Entries 4, 5 and 6 explore the effect of reducing the time step upon the operational scheme. Only when the time step is reduced to 300 secs do we find the computation to be stable for appreciable wind speeds (up to 30 m/sec).

Entry #7 shows the result of using 2nd order finite differences with the operational value of diffusion and the 400 sec time step. The result is stable for wind speeds up to 40 m/sec.

If one were to judge by results to this point, it appears that second order differences should be used rather than reducing the time step. However, we decided to evaluate the effect of only using second-order accurate approximations for the gravity wave leaving advection as fourth order accurate. The rationale for this is based on the assumed meteorological unimportance of the gravity wave component of the numerical solution.

Entry #8 shows that in the absence of diffusion this approach is successful. Comparison with entries 2 and 3 is suggested.

When diffusion was added, the allowable wind speed for stability was cut back to 30 m/sec as shown by entry #9.

We tested this approach by using conditions appropriate for latitude 20°N and found the results to be stable for wind speeds up to 50 m/sec as shown by entry 10.

In entries #11 and #12, we show that, by reducing the time step by 10% to 360 sec and by using second-order accurate approximations of the gravity wave terms while retaining fourth order accuracy for advection, one may obtain a wholely satisfactory linear stability constraint.

| Entry # | DT | VMAX | DX | K | GRAV | ADV |
|---|---|---|---|---|---|---|
| 1 | 400 | ---- | 120 | 1.8E6 | 4th | 4th |
| 2 | 400 | 10 | 120 | 0 | 4th | 4th |
| 3 | 400 | 70 | 120 | 0 | 2nd | 2nd |
| 4 | 360 | ---- | 120 | 1.8E6 | 4th | 4th |
| 5 | 330 | 5 | 120 | 1.8E6 | 4th | 4th |
| 6 | 300 | 35 | 120 | 1.8E6 | 4th | 4th |
| 7 | 400 | 40 | 120 | 1.8E6 | 2nd | 2nd |
| 8 | 400 | 70 | 120 | 0 | 2nd | 4th |
| 9 | 400 | 30 | 120 | 1.8E6 | 2nd | 4th |
| 10 | 400 | 50 | 20°N | 2.2E5 | 2nd | 4th |
| 11 | 360 | 50 | 120 | 1.8E6 | 2nd | 4th |
| 12 | 360 | 70 | 20°N | 2.2E5 | 2nd | 4th |

TABLE: RESULTS AND COMPUTATIONS

DT: time step sec
VMAX maximum wind speed for stability m/sec
DX grid size km
K diffusion coefficient $m^2$/sec
GRAV order of accuracy of gravity wave
ADV order of accuracy of advection

## 5. CONCLUSION AND RECOMMENDATION

We have found that the operational LFM numerical method is only marginally stable at low latitudes. The difficulties encountered early in March 83 were probably related to the violation of the linear stability criterion.

The precision of the analysis presented here depends in part on the true phase speed of the external gravity mode. We may have erred on the side of conservatism by setting C equal to 330 m/sec. In any event there is good reason to believe that the theoretical results presented here are in good agreement with our recent experience.

It is recommended that the gravity-wave terms be approximated using second-order accurate finite difference approximations. We should retain fourth order accuracy on advective terms at least in the momentum, thermodynamic and water-vapor conveyance equations.

The case in which the LFM completely failed earlier in March should be rerun with the proposed change in difference approximations.

To instill even greater confidence that this problem is put behind us, I'd further recommend that the time step be reduced by 10% to 360 sec. This reduction can be well afforded on the Cyber 205 and would be reasonable even on the 360/195's.

## REFERENCES

Brown, J. A., Jr., and K. A. Campana, 1978: In economical time-differencing system for numerical weather prediction. MWR, 106:8, pp. 1125-1136.

Schoenstadt, A. L., and R. T. Williams, 1976: The computational stability properties of the Shuman pressure gradient averaging technique. J. Comp. Phys., 21, pp. 166-177.