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An Imaginary Instability in Data Assimilation with Normal Mode Initialization

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Abstract

Small numerical error in the calculation of normal modes can result in zonal modes not being exact conjugate pairs. This causes the nonlinear normal mode initialization to produce zonal model fields with a small amplitude imaginary component, which should be identically zero. These small amplitude imaginary fields can become unstable due to inconsistency in the use of these unreal fields in the normal mode initialization. An example is given of such instability in the NMC assimilation system.

1. Introduction

The basic fields of a numerical weather prediction model should be real. However, some transforms of model fields such as Fourier coefficients, spectral coefficients, and normal mode projections are complex. Since the basic fields are real, certain transforms are purely real. Thus the imaginary portions of such transform variables are sometimes used or not used in numerical routines depending on factors of convenience, efficiency, and accuracy. That is, if something is assumed to be identically zero it may be included or excluded from certain calculations with no effect on the result.

With a spectral forecast model, the spectral coefficients are complex, but the zonal coefficients should be identically real. Furthermore, the zonal gravity modes should occur in exact conjugate pairs. For example, if there is an eastward moving gravity mode of frequency ω, there will be a corresponding westward moving mode of frequency $-\omega$. With the NMC spectral model, some initialization codes use the imaginary portion of zonal coefficients and some do not. This is fine if the zonal coefficients are purely real, but if by some error they are not real, there can be problems.

After several days of assimilation, the NMC spectral model developed sizable imaginary zonal divergence. This divergence grew unstably and eventually adversely affected the real portion of the spectral analysis. All model fields showed small imaginary zonal coefficients after initialization, but it was only the imaginary divergence that became large. Here we describe the problem as it was observed, the source of the imaginary zonal divergence, and the mechanism for its instability.

2. The Observed Instability Problem

During the period 12Z, 28 October 1982 to 12Z, 7 November 1982 the global data assimilation system (GDAS), described in McPherson et al. (1979) and Kistler and Parrish (1982), had cycled continuously for 10 days without interruption. Near the end of this period certain analyzed fields began to exhibit nonmeteorological features. Specifically, the upper tropospheric wind fields suddenly increased in magnitude, with no corresponding change in the respective height fields. Examination of the RMS divergence from these analyzed fields revealed values approaching unity, when in fact the RMS values should be of the same order of magnitude as the fields themselves, approximately 10^{-5} sec^{-1}. The spectral forecast model showed extremely large RMS mass-field tendencies, which was consistent with the large RMS divergence. However, the RMS vorticity and divergence tendencies were surprisingly normal. In addition, when the divergence fields were examined on the model's grid they were found to be normal, which was contrary to the extremely large RMS divergence discovered earlier.

Closer examination of the spectral coefficients themselves revealed that the imaginary part of the zonal coefficients were nonzero and of the order unity. The imaginary part of the zonal coefficients should in fact be identically zero in order to produce spatial fields that are real. This explains why the gridded fields and divergence and vorticity tendencies appeared normal. The mapping routine assumes real zonal coefficients and therefore ignored the imaginary portions. Similarly, the model's calculation of vorticity and divergence tendencies accessed only the real portion of the zonal coefficients, thereby producing reasonable tendencies.

Further investigation revealed that given real zonal coefficients the model's normal mode initialization step will produce small nonzero imaginary zonal coefficients. With further use of normal mode initialization, the imaginary zonal divergence begins to increase and eventually was approximately doubling in magnitude with each use of initialization. On the other hand, the analysis step returned the identical imaginary zonal coefficients as its first guess, and the model's six hour forecast caused a slight reduction in its magnitude. This indicates that the analysis code does not specifically deal with the imaginary portion of the zonal coefficients, while the initialization and forecast model use the entire coefficient, having made the assumption that the imaginary portion is identically zero. After ten days of cycling, the imaginary coefficients were large enough to contaminate the real fields. What remains to be explained is how the initialization caused the coefficients to become nonzero and why it produced a systematic increase in their magnitude.

3. Explanation of the Instability

To understand this problem, we examine linearized model equations for the zonal fields in spectral form as used in the normal mode initialization; further details on this subject can be found in Sela (1982). The linearized divergence tendency for zonal coefficient D_n^0 can be written

$$\frac{\partial D_n^0}{\partial t} = 2\Omega\left(\frac{n+1}{n}\epsilon_n^0 \zeta_{n-1}^0 + \frac{n}{n+1}\epsilon_{n+1}^0 \zeta_{n+1}^0\right) + \frac{n(n+1)}{a^2} P_n^\ell \quad (1)$$

where n is an integer from 0 to 30, Ω is the Earth's angular velocity, ζ is vorticity, P is a massfield variable used in the initialization

$P = \phi + RT_0 \log P_s$, and $\epsilon_n^\ell = \left((n^2-\ell^2)/(4n^2-1)\right)^{1/2}$.

For simplicity we write (1) as

$$\frac{\partial \underset{\sim}{D}}{\partial t} = F\underset{\sim}{\zeta} - \nabla^2 \underset{\sim}{P} \quad (2)$$

where $\underset{\sim}{D} = (D_0^0, D_1^0, \cdots D_{30}^0)^T$, F is a tridiagonal matrix, with $\underset{\sim}{P}$ and $\underset{\sim}{\xi}$ defined like $\underset{\sim}{D}$. The remaining linear equations used after projecting onto vertical modes are

$$\frac{\partial \underset{\sim}{\xi}}{\partial t} = -F\underset{\sim}{D} \tag{3}$$

$$\frac{\partial \underset{\sim}{P}}{\partial t} = -gh\underset{\sim}{D} \tag{4}$$

where h is the equivalent depth for the given vertical mode.

The normal modes of linear equations (2)-(4) are somewhat simpler than nonzonal modes. For the above linearization, Ballish (1980) has shown that Machenhauer's (1977) normal mode initialization adjusts the divergence equivalent to

$$\delta \underset{\sim}{D} = (F^2 - gh\nabla^2)^{-1} (F\dot{\underset{\sim}{\xi}} - \nabla^2 \dot{\underset{\sim}{P}}) \tag{5}$$

where $-gh\nabla^2$ is assumed to be a diagonal matrix with elements $n(n+1)gh/a^2$, and $\delta\underset{\sim}{D}$ is the change to be made in the divergence of a given vertical mode. Equation (5) is very similar to the quasi-geostrophic omega equation in that we are requiring divergence or vertical velocities to make the vorticity and mass-field tendencies geostrophically related.

We can now examine how the initialization introduced imaginary fields. The initialization step can introduce imaginary changes even when it has real fields and tendencies to start with. This error is due to the calculated zonal normal modes not coming in exact conjugate pairs. That is, there should be pairs of eastward and westward moving gravity modes with the corresponding eigenvalues and vectors being complex conjugates of each other. The modes are calculated in double precision and then truncated to single precision for later use in the initialization. The resulting single precision frequencies are in exact conjugate pairs, but the eigenvectors have slight error. Some modal spectral coefficients with relatively small amplitude show nonconjugate agreement in the fifth decimal place. The modes are quite accurate for most purposes but do have slight error in their low amplitude coefficients. As a result of this error, the initialization step makes some imaginary changes in the eastward and westward moving zonal modes that should add to give a real result but instead give some results with a small imaginary component.

Now that some imaginary divergence has been created, we explain how it becomes unstable and grows. The imaginary divergence leads to some model tendencies being imaginary. One part of the initialization code calculates nonlinear tendencies as well as tendencies involving the Coriolis effect. Another code calculates various linear tendencies not included in the first code. The first code calculates tendencies using the fields on the model's grid and does not use the imaginary zonal coefficients. However, for the linear tendencies, the imaginary zonal coefficients are treated the same as other coefficients. The imaginary divergence has no effect on the divergence and vorticity tendencies which remain purely real, but the massfield tendency has an imaginary component. Thus when the initialization adjusts the zonal divergence according to (5) we have, with $\underset{\sim}{D}_I$ defined to be the vector of the imaginary part of the zonal divergence,

$$\delta \underset{\sim}{D}_I = (F^2 - gh\nabla^2)^{-1}(gh\nabla^2 \underset{\sim}{D}_I) \tag{6}$$

$$\delta \underset{\sim}{D}_I = -(I - (gh\nabla^2)^{-1}F^2)^{-1} \underset{\sim}{D}_I \tag{7}$$

where we have taken the liberty to write $(\nabla^2)^{-1} = -a^2/n(n+1)$ except for $n=0$ where $(\nabla^2)^{-1}$ is taken to be zero. This is acceptable because the initialization code does not alter mean fields with $n=0$. For modes with large h and n, $\delta D_I \approx -D_I$, or the initialization is getting rid of the imaginary divergence. However, for modes with smaller h and n , δD_I may be larger than D_I itself. The problem is complicated in that F is a tridiagonal matrix so that if $\underset{\sim}{D}_I$ is 0.0 except for $D_n^o \neq 0.0$, then (7) leads to $\delta \underset{\sim}{D}_I$ having nonzero coefficients $D_{n'}^o$ where n' is of the same parity as n .

This hypothesis for the instability was tested by performing several iterations of the initialization with data which had real divergence except that D_9^o was imaginary. After one iteration, D_9^o was reduced noticeably, but now several other coefficients of D have significant imaginary values. With further interations, the overall amplitude of imaginary divergence increases and spreads its amplitude over more coefficients.

Note that if $-F\underset{\sim}{D}_I$ had been included in $\dot{\underset{\sim}{\xi}}$ used in (5), the initialization would have simply gotten rid of $\underset{\sim}{D}_I$.

By the time that the RMS imaginary divergence was of the order 1 sec^{-1}, we also noticed some small amplitude nonmeteorological appearing patterns in our height and wind analysis maps. This is believed to be caused by the very large imaginary divergence having some small effect on other real fields. The details of this effect have not been investigated; however, it may simply be due to numerical round off error associated with calculations using the large imaginary divergence.

4. Conclusions

We have seen that small numerical error in the calculation of normal modes can lead to small imaginary error in the zonal fields. The inconsistent use of imaginary zonal fields in calculating tendencies for normal mode initialization can then lead to instability. Anyone using normal mode initialization should take care that they do not have similar problems.

For anyone using the spectral coefficients from any history tapes from our GDAS, they should not use the imaginary portion of zonal coefficients from our analyses, initialized analyses, or forecasts. However, as of 1200 GMT 17 November 1982, our analysis set to zero the imaginary portion of all zonal fields.

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