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OFFICE NOTE 282

VERTICAL INTERPOLATION OF HEIGHTS AND TEMPERATURES FOR MODEL INPUT/OUTPUT

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I. Introduction

Heights and temperatures must be vertically interpolated since they are not observed, reported, analyzed, forecasted, and verified all at the same pressures. Since the accuracy of the interpolation can affect the overall skill of the analysis/forecast system it is good to have some well-defined principles and procedures in hand to perform the interpolation. It is taken as a basic tenet that consistency is important in all procedures used. This note will deal with both the basic principles underlying the proposed vertical interpolation method and the derivation of a concrete set of accurate procedures.

The problem consists in going from mandatory (and possibly significant) level data to a model vertical coordinate and back with minimum error. The fact that a forecast has been performed by the model in the interum is irrelevant.

II. Basic Principles

Derivations from a hydrostatic balance are not definable from radiosonde observations because of reduction methods. The deviations can be considered to have a normal distribution with zero mean. Therefore, it is justified to use the hydrostatic equation to relate temperature and height differences. It can be written

$$g\,dz = -RT\,d\ln p \equiv -RT\,dP \qquad \text{(II.1)}$$

In integrated form

$$g\Delta z = \int_{z}^{z+\Delta z} dz = -R\int_{P}^{P+\Delta P} T\,dP \qquad \text{(II.2)}$$

It is also noted that since all NMC operational forecase models are hydrostatic, hydrostatically consistent heights and temperatures provide needed model input. Furthermore, the reported heights are themselves obtained from the radiosonde reports by hydrostatic integration of the temperature field.

Model input is customarily provided by input at mandatory levels only, so that the height field includes the influence of all temperatures, but not vice versa. Therefore, we may regard the height field to be most representative of the overall structure of the atmosphere. For these reasons, this note will regard the height field to be of more fundamental nature than temepratures. In essence, this means that all vertical interpolation will be interpolation of the height field, with temperatures determined hydrostatically from it.

The weight of a column of air, expressed as height differences, is related to the virtual temperature of the air, not the dry air temperature. Thus, the temperature in (II.1-2) is the virtual temperature. It is given approximately by

$$T_V \approx (1.+0.61q)T \qquad \text{(II.3)}$$

where T_V is the virtual temperature, T is the temperature, and q is the specific humidity. In all that follows, temperature will be used to mean virtual temperature. It is noted that none of the NMC operational models properly takes the effect of moisture into account for input and output. This should be an area for future consideration.

One test of a vertical interpolation is that it should produce no errors when going from mandatory level heights to a model with levels at the mandatory pressures and back to mandatory levels. This requirement will eliminate some schemes which may be good in an RMS sense but do not pay proper attention to the discrete structure of the problem. Reversibility and consistency are principles which will guide the development of suitable interpolation schemes.

III. Design of a Method

There are a large number of methods that would meet the design criteria set up in the last section. They are limited, however, to methods which have no error at the input levels, i.e., co-location polynomials are suitable, but cubic splines are not. This note will consider only fairly simple forms--in which the height varies quadratically with a vertical variable. The forms examined are

$$\begin{Bmatrix} T \\ \theta \end{Bmatrix} \text{ varies linearly with } \begin{Bmatrix} p \\ \ln p \\ \pi \\ z \end{Bmatrix} \tag{III.1}$$

where one choice on the left is taken with one choice on the right. Variables have the usual definition.

The accuracy of an interpolation scheme may be judged by the error in going from mandatory levels to model structure and back. In this process it is necessary to pay attention to the model structure, including vertical staggering of variables and required thermodynamic variable. But it is not needed to actually make a forecast. In fact, doing so would only add error. It is also not necessary (indeed, it is not desirable) for the interpolation method to use the model's definition of the mid-layer coordinate (if it exists).

IV. Linear Equations

The next section will give the justification for using the method presented in this section. It is shown that the most accurate of the equations (III.1) is

$$T \propto \ln p \tag{IV.1}$$

The Appendices A-H will give the equations for all the 8 methods since they may at times be useful. Here, only the equation for (IV.1) will be considered.

Let the temperature variation over some pressure range be given by

$$T = \overline{T} + b(P - \overline{P}) \tag{IV.2}$$

where $P \equiv \ln p$. Substitution into the hydrostatic equation, (II.2) gives

$$\Delta z = -\frac{R}{g}\int_{P}^{P+\Delta P} (\overline{T} + b(P-\overline{P}))\,dP$$

$$= -\frac{R}{g}(\overline{T} - b\overline{P})\Delta P - \frac{Rb}{g}\left(P + \frac{\Delta P}{2}\right)\Delta P \tag{IV.3}$$

solving for $\overline{T}$ gives

$$\overline{T} = b\overline{P} - b\left(P + \frac{\Delta P}{2}\right) - \frac{g}{R}\frac{\Delta z}{\Delta P} \tag{IV.4}$$

Consider application of (IV.3-4) for input from mandatory level data. On input to a numerical model, the heights are known. Also, the lapse rate may be estimated between mandatory levels to give an estimate of $b = dT/d\ln p$. Therefore, (IV.4) gives the relationship between $\overline{T}$ and $\overline{P}$. A natural choice is

$$\overline{T} = -\frac{g}{R}\frac{\Delta z}{\Delta P} \tag{IV.5}$$

and

$$\overline{P} = P + \frac{\Delta P}{2} \tag{IV.6}$$

Note that this choice gives a usual form for the relationship between a "mean" temperature, $\overline{T}$, and the height difference and also eliminates the lapse rate from the forms for $\overline{T}$ and $\overline{P}$. A similr procedure is followed in the Appendices to find the relationship between the "mean" thermodynamic variable and "mean" coordinate for the other forms. It is stressed that they can never be chosen independently.

Now, consider the interpolation of heights within a layer where (IV.2) holds. Integration from the base of the layer at (Z_0, P_o) to the ln (pressure) = P gives

$$z = z_0 - \frac{R}{g}(\bar{T} - b\bar{P})(P - P_0) - \frac{Rb}{g}\left(\frac{P+P_0}{2}\right)(P - P_0) \tag{IV.7}$$

This formula can be used to provide model heights at given pressures. Details of the actual procedure will be given later.

An equation is also needed to give the pressure at a known height. It is used to find the surface pressure for the model terrain height. Figure 1 shows the data arrangement.

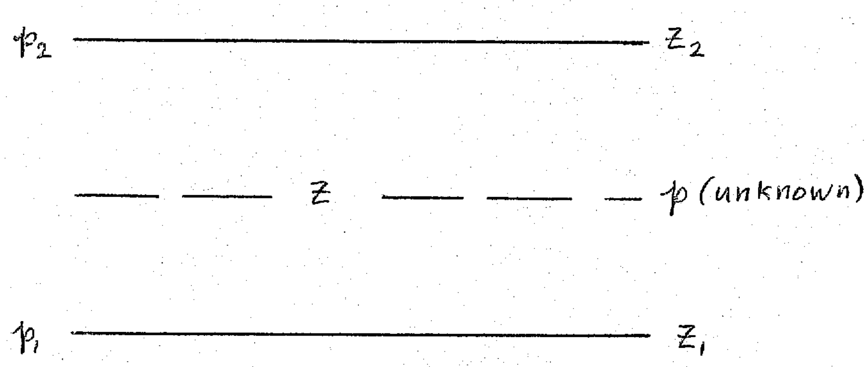


Figure 1. Data arrangement for solution for pressure at given height.

In this case

$$\bar{T} = -\frac{g}{R}\,\frac{z_2 - z_1}{P_2 - P_1}, \quad \bar{P} = \frac{P_1 + P_2}{2} \tag{IV.8}$$

Define $\delta z = z - z_1$, $\delta P = P - P_1$ and $\Delta P = P_2 - P_1$. (IV.9)

Then the hydrostatic equation gives

$$\delta z = -\frac{R}{g}\left\{\bar{T} + \frac{b}{2}(\delta P + \Delta P)\right\}\Delta P \tag{IV.10}$$

Straight-forward solution for δP gives

$$\delta P = \frac{1}{b}\left\{-\left(\bar{T} + \frac{b}{2}\Delta P\right) \pm \left[\left(\bar{T} + \frac{b}{2}\Delta P\right)^2 - 2b\frac{g}{R}\delta z\right]^{1/2}\right\} \qquad \text{(IV.11)}$$

But this form is unsuitable since b can be small. Instead, (IV.10) is solved by a different application of completion of squares (Shuman and Hovermale, 1968). The result is

$$\delta P = \frac{\delta z}{-\frac{R}{2g}\left(\bar{T} + \frac{b}{2}\Delta P\right) - \sqrt{\left[\frac{R}{2g}\left(\bar{T} + \frac{b}{2}\Delta P\right)\right]^2 - \frac{Rb}{2g}\delta z}} \qquad \text{(IV.12)}$$

It has been stated that it is important to use the proper $\bar{P}$ with $\bar{T}$. Letting $\epsilon(\bar{P})$ be the error in $\bar{P}$, then the error in the height difference for a layer of ln-pressure thickness ΔP is

$$\epsilon(\Delta z) = \frac{R}{g} b \Delta P \cdot \epsilon(\bar{P}) \qquad \text{(IV.13)}$$

For errors from using mean ln (pressure) based upon p or z, the error in height should be from less than a meter to several meters at high altitudes.

V. Determination of Best Linear Law

It is believed that the temperature law (III.1) which best fits the atmosphere will also be the best law to use for the vertical interpolation. This section examines the determination of that best fit. Mandatory level data are used for the eastern U.S. for all reporting radiosonde stations and all possible pressure levels, a linear interpolation is made between alternate pressure levels. I.e., 1000 mb and 700 mb temperatures are interpolated to 850 mb; 850 and 500 mb temperatures are interpolated to 700 mb; etc. The interpolated temperatures are compared with the reported temperatures and mean error, RMS error and error frequency distributions are generated. Figures 2 -9 show the mean and RMS errors as a function of pressure for the 8 temperature laws. Those using temperature (rather than potential temperature) are all reasonably close and are best. The best overall is $T \propto \ln p$ with $T \propto Z$ a close second. The largest errors are near the tropopause and ground.

Figures 10-11 show the frequency distribution of the errors, averaged with respect to pressure. Again, $T \propto \ln p$ is shown to be most accurate. The error pattern is skewed toward positive values because of the positive errors near the tropopause and ground. All the methods using potential temperature, on the other hand, have significant negative and positive errors. These methods have extremely large RMS errors above 20 mb.

VI. Recommended Procedures

Figure 12 shows data arrangement for mandatory levels and both staggered and unstaggered model structure. The method of interpolation will differ somewhat for staggered and unstaggered model structure. It will also differ slightly between models using T and those using θ as the thermodynamic variable.

Preparation for interpolation begins with solving for mean temperature (T) or potential temperature (θ) for the layers between mandatory levels, using the appropriate form of the hydrostatic equation. The specification of the temperature in each layer is completed by specifying b, the lapse rate. The LFM analysis analyzes temperatures as well as heights so that the lapse is directly specified. More usually, a temperature difference across two layers needs to be used, where the temperatures have been determined hydrostatically. At this point, a distinction must be made between models with staggered and unstaggered variables in the vertical. However, in any case, the height interpolation will be to model levels. (Levels are defined to be constant sigma surfaces which bound layers. Usually, the primary definition of sigma-values is at levels.) The interpolation formulas are used to integrate the height from the ground upward to each sigma-level. When parts of more than one mandatory data layer are between two sigma-levels, then the temperature profile appropriate to each part must be used.

A. Staggered grid

For staggered T (or θ) and Z, the interpolated level heights are used, together with the model hydrostatic equation to obtain the model thermodynamic variable. On output the model heights are first obtained from the model thermodynamic variable and model form of the hydrostatic equation. Interpolation from the model heights to mandatory level heights is accomplished using only interpolation forms to define mean temperature and mean layer pressure.

B. Unstaggered

In this case the model heights are in the middle of layers, but the interpolation still begins by interpolating heights to model levels. From the level heights, determine the layer temeprature by using the interpolation form of the hydrostatic equation. If the wrong thermodynamic variable is obtained (θ versus T), then use the interpolation-defined mean coordinate to make the conversion. Model heights, in layers, are obtained when needed by using the model hydrostatic equation.

On output the process is reversed. First, the model thermodynamic variable is converted, if necessary, to the form used in the interpolation. The interpolation-defined mean coordinate is used. Lapse rates are defined and interpolation forms are used to interpolate to mandatory levels. As on input, the heights are obtained by integration of the temperature profile.

C. Determination of lapse rate

The effect of the lapse rate on the thickness of a layer is of second order in ln (pressure) (see IV.10). Therefore, the specification of the lapse is not critical, but the best estimate should be used. The data and model structure will suggest the appropriate method. Figure 13 shows three methods presently used somewhere in NMC codes. Figure 13a shows the easiest situation. $\overline{T}$ is determined (hydrostatically) from Z's and b's are determined from the T's. Figure 13b shows the more usual situation where only Z's are input. Then $\overline{T}$'s are obtained from the Z's and the b's are obtained from the $\overline{T}$'s by finite differences. In Figure 13c, p** is the tropopause level, so we will want to extrapolate temperatures from below and above. As for Figure 13b, Z's determine $\overline{T}$'s.

D. Vertical interpolation of residuals in the optimum interpolation analysis

It will be shown in this section that the same procedure can be used for vertical interpolation of O/I height residuals as for the total height field. First, it will be shown that the hydrostatic equation is the same, and second it will be shown that the temperature law for the residuals is the same as for the total field, but with the lapse rate determined by the residuals.

The hydrostatic equation is written (for a layer)

$$\Delta z = -\frac{R}{g}\overline{T}\Delta P \qquad \text{(VI.1)}$$

For an updated layer

$$\Delta z + \delta z = -\frac{R}{g}(\overline{T} + \delta\overline{T})\Delta P \qquad \text{(VI.2)}$$

The pressure thickness remains the same and the height increment δZ, determines the change in the mean temperature, $\delta\overline{T}$. Subtracting (VI.1) from (VI.2) gives the hydrostatic equation for the residuals.

$$\delta z = -\frac{R}{g}\delta\overline{T}\Delta P \qquad \text{(VI.3)}$$

Now, consider the temperature variation. Originally, it is assumed to be

$$T = \overline{T} + b(P - \overline{P}) \qquad \text{(VI.4)}$$

with

$$\bar{T} = -\frac{g}{R}\frac{\Delta Z}{\Delta P}\ ,\ \bar{P} = P + \frac{\Delta P}{2}\ \text{and}\ \ b = \frac{dT}{dP} \tag{VI.5}$$

After update

$$T + \delta T = (\bar{T} + \delta\bar{T}) + b'(P - \bar{P}) \tag{VI.6}$$

where

$$b' = \frac{d(T + \delta T)}{dP} \tag{VI.7}$$

Subtracting (VI.4) from (VI.6) gives the temperature law for the residuals

$$\begin{aligned} \delta T &= \delta\bar{T} + (b' - b)(P - \bar{P}) \\ &= \delta\bar{T} + \frac{d(\delta T)}{dP}(P - \bar{P}) \end{aligned} \tag{VI.8}$$

Equations (VI.3), (VI.8) show that the residuals follow the same equations as the full fields, only the correct lapse rate must be used in (VI.8). Therefore, the methods recommended elsewhere in this note equally apply to the vertical interpolation of O/I analysis residuals to given pressure levels.

E. SHUELL pressure reduction

The SHUELL pressure reduction method is a combination of methods of F. Shuman and J. Newell. It is described in Technical Procedures Bulletin No. 57. The methods uses the assumption that temperature varies linearly with height. The uniqueness of the method lies in the way the mean layer temperature and lapse rate are specified. The LFM and Spectral models use slightly different formulations. The method as described here uses the equations of Appendix D. Refer to Figure 14 for notation. In the LFM, the layer between Z_1 and Z_s is the boundary layer; in the Spectral model, Z_1 and Z_s are the same height.

First, define the mean temperature for the lowest sigma-layer used (Z_1 to Z_2).

$$\bar{T} = -\frac{\Gamma}{2}(Z_2 - Z_1)\left(\frac{1+\alpha}{1-\alpha}\right) \qquad \text{(VI.9)}$$

where

$$\alpha = \left(\frac{p_2}{p_1}\right)^{-\frac{R\Gamma}{g}} \qquad \text{(VI.10)}$$

and Γ = -.0065 deg/m is the standard atmosphere lapse rate. Next, extrapolate temperature downward, with lapse rate Γ, to Z_s and Z=0.

$$T_s = \bar{T}_1 - \Gamma(\bar{Z}_1 - Z_s) \qquad \text{(VI.11)}$$

$$T_o = \bar{T}_1 - \Gamma\bar{Z}_1 \qquad \text{(VI.12)}$$

Next, the lapse rate below ground is specified to model the procedure followed at observing stations (see Bigelow, 1902). The lapse rate is determined from T_s and T_o. However, the provisional value of T_o from (VI.12) may be changed first as follows:

1) If $T_o \leq 290.66$K, then no change to T_o
2) If $T_s > 290.66$K, then $T_o = 290.66 - .005*(T_s - 290.66)^2$
3) If $T_o > 290.66$K and $T_s < 290.66$K, then $T_o = 290.66$K.

The lapse rate is calculated as

$$b = \frac{T_s - T_o}{Z_s} \qquad \text{(VI.13)}$$

The mean layer (Z=0 to Z_s) temperature and height are defined by

$$\bar{T}_o = -\frac{b}{2} Z_o\left(\frac{1+\beta}{1-\beta}\right) \qquad \text{(VI.14)}$$

$$\beta = \left(\frac{p_s}{p_{MSL}}\right)^{-\frac{Rb}{g}} \qquad \text{(VI.15)}$$

$$\bar{Z}_o = \frac{1}{2} Z_s \qquad \text{(VI.16)}$$

If b=0, then

$$z_M = -\frac{R}{g}\overline{T}_o \ln\left(\frac{p_M}{p_{MSL}}\right) \tag{VI.17}$$

Otherwise

$$z_M = \left(\frac{\beta-1}{b}\right)\left(\overline{T}_o - b\overline{z}_o\right) \tag{VI.18}$$

The equations used by the LFM and Spectral models are approximations to (VI.18). The Spectral model equation is

$$z_M = z_s + \frac{\frac{R}{g}T_s \ln(p_s/p_M)}{1-\frac{1}{2}b\frac{R}{g}\ln(p_s/p_M)} \tag{VI.19}$$

while the LFM uses (VI.17) with $\overline{T}_o$ replaced by $1/2(T_o+T_s)$.

VII. Methods Used by Operational Models

A short sketch of the vertical interpolation used by the NMC models will be listed. To a reasonable degree, the recommendations of this note are already followed, but no model has complete consistency.

A. LFM-INI (subroutine PTOSIG)

Surface pressure - interpolation to ground height assumes T lnp.

Heights at sigma-levels - interpolation assumes $T \propto \ln p$. The same form is used to get mandatory level heights on output.

Potential temperatures - generally obtained by θ, π form of hydrostatic equation from heights, but the lowest layer uses special technique designed by J. Stackpole to utilize the surface temperature. The boundary layer θ is an interpolation between the surface temperature and the first mandatory level above, provided that $Z_s<Z_{850}$. The method assumes $T \propto \ln p$. The $\overline{P}$ definition is implicit in the choice of $\tilde{\theta}$.

B. LFM-POST (subroutine SIGTOP)

Mid-layer π - $\hat{\pi} = \Delta\pi/K\,\Delta\ln p$. This is inconsistent and should be changed to $\pi(\overline{P})$.

Mid-layer pressure - $p(\hat{\pi})$. See above.

Level temperatures - interpolated using $T \propto \ln p$, but with extrapolation to tropopause. (Method (c) described in section VI.C.)

Level heights - by hydrostatic equation in θ, π form.

Tropopause temperature - average of extrapolations from above and below.

FD pressures - the method uses $\hat{\pi}$ to define mid-layer and the assumption that $\theta \propto \pi$. (This is a bad method and should be changed.)

FD winds, potential temperatures - interpolated linearly with respect to π.

Mandatory level winds, relative humidity, temperatures - interpolated linearly with respect to lnp (but with mid-$\pi = \hat{\pi}$). A standard lapse rate of 6.5×10^{-3} deg/m is used below the middle of the lowest layer. (The dew point is calculated according to the formula in Office Note 36.)

Mandatory level heights - interpolated assuming $T \propto \ln p$. The lapse rates are specified from the level temperatures (specific as discussed above). The SHUELL pressure reduction technique is used below ground. It uses $T \propto Z$ with specified lapse rates.

C. Spectral INI (subroutine PTOSIG)

Sigma-level heights - interpolation assumes $T \propto \ln p$.

Sigma-layer temperatures - solved from hydrostatic equation in T, lnp form, using level heights. The $\overline{P}$ definition is implicit is the hydrostatic form uses. Note! The level heights are not used by the model. They are only used to obtain the temperatures as described above.

Surface pressure (subroutine GETPS) - interpolated assuming $T \propto \ln p$.

D. Spectral POST (subroutine SIGTOM)

Mid-layer sigma - corresponds to $\hat{\pi} = \Delta \pi / K \Delta \ln p$.

Level heights - solved from hydrostatic equation in T, lnp form.

Mid-layer π - $\hat{\pi}$.

Mid-layer lnp - ln of pressure corresponding to $\hat{\pi}$.

Mid-layer p - $\overline{p}$ (arithmetic mean) (not p($\hat{\pi}$)).

Level temperatures - interpolation or extrapolation assuming T $\propto$ lnp.

Level temperature (also RH, U, V, ω) - linear interpolation with respect to lnp.

Mandatory level heights - interpolation using T $\propto$ lnp except below ground where SHUELL technique is used.

Level temperatures - are recomputed using T. Flattery's subroutine LOWTMP if any levels are below ground.

Appendix A. $T \propto \ln p$

All the Appendices have the same form, each dealing with a different temperature law. The parts are:

1. the temperature law

2. the hydrostatic equation, integrated through a layer

3. the equation relating the mid-layer coordinate and mid-layer temperature

4. the equations for a reasonable choice of mid-layer coordinate and mid-layer temperature

5. the hydrostatic equation, integrated through a partial layer, using the choice for mid-layer coordinate and temperature

6. the method to obtain the pressure at a given height is presented

<u>temperature law</u>

$$T = \bar{T} + b(P - \bar{P}) \qquad P \equiv \ln\ (\text{pressure}) \tag{A.1}$$

<u>hydrostatic equation for layer</u>

$$\Delta z = -\frac{R}{g}(\bar{T} - b\bar{P})\Delta P - \frac{Rb}{g}\left(P + \frac{\Delta P}{2}\right)\Delta P \tag{A.2}$$

<u>$\bar{T},\bar{P}$ equation</u>

Solving (A.2) for $\bar{T}$ gives

$$\bar{T} = b\bar{P} - b\left(P + \frac{\Delta P}{2}\right) - \frac{g}{R}\frac{\Delta z}{\Delta P} \tag{A.3}$$

<u>choice of $\bar{T},\bar{P}$</u>

The following choice agrees with usual practice and eliminates b from the expressions.

$$\bar{T} = -\frac{g}{R}\frac{\Delta z}{\Delta P} \tag{A.4}$$

$$\bar{P} = P + \frac{\Delta P}{2} \tag{A.5}$$

hydrostatic equation for partial layer

Refer to Figure 15 for notation

$$z = z_o - \frac{R}{g}(\bar{T} - b\bar{P})(P - P_o) - \frac{Rb}{g}\left(\frac{P + P_o}{2}\right)(P - P_o) \tag{A.16}$$

pressure at given height

Refer to Figure 16 for notation

$$\delta P = \frac{\delta z}{-\frac{R}{2g}(\bar{T} + \frac{b}{2}\Delta P) - \sqrt{\left[\frac{R}{2g}(\bar{T} + \frac{b}{2}\Delta P)\right]^2 - \frac{Rb}{2g}\delta z}} \tag{A.17}$$

Appendix B. $T \propto p$

temperature law

$$T = \bar{T} + b(p - \bar{p}) \tag{B.1}$$

hydrostatic equation for layer

$$\Delta z = -\frac{R}{g}(\bar{T} - b\bar{p})\Delta P - \frac{R}{g} b \Delta P \qquad (P \equiv \ln p) \tag{B.2}$$

$\bar{T}, \bar{p}$ equation

Solving (B.2) for $\bar{T}$ gives

$$\bar{T} = -\frac{g}{R}\frac{\Delta z}{\Delta P} + b\left(\bar{p} - \frac{\Delta p}{\Delta P}\right) \tag{B.3}$$

choice for $\bar{T}, \bar{p}$

One natural choice is

$$\bar{T} = -\frac{g}{R}\frac{\Delta z}{\Delta P} \qquad (\text{as before}) \tag{B.4}$$

$$\bar{p} = \frac{\Delta p}{\Delta P} \tag{B.5}$$

hydrostatic equation for partial layer

$$z = z_0 - \frac{R}{g}(\bar{T} - b\bar{p})(P - P_0) - \frac{R}{g} b (p - p_0) \tag{B.6}$$

pressure at a given height

A numerical solution must be used. One method follows. Let

$$P' = P_0 - \frac{g}{R\bar{T}}(z - z_0) \tag{B.7}$$

and

$$p^n = \exp(P^n) \tag{B.8}$$

$$P^{n+1} = P_0 - \frac{\frac{g}{R}(z - z_0) + b(p^n - p_0)}{\bar{T} - b\bar{p}} \tag{B.9}$$

For b= .04 deg/mb this scheme converges to 8 decimal places in about 9 scans.

Appendix C. $T \propto \pi$

temperature law

$$T = \bar{T} + b(\pi - \bar{\pi}) \tag{C.1}$$

hydrostatic equation for layer

$$\Delta z = -\frac{R}{g}(\bar{T} - b\bar{\pi})\Delta P - \frac{Rb}{Kg}\Delta\pi \tag{C.2}$$

$\bar{T},\bar{p}$ equation

Solving (C.2) for $\bar{T}$ gives

$$\bar{T} = b\bar{\pi} - \frac{g}{R}\frac{\Delta z}{\Delta P} - \frac{b}{K}\frac{\Delta\pi}{\Delta P} \tag{C.3}$$

choice for $\bar{T},\bar{p}$

One natural choice for $\bar{T},\bar{\pi}$ satisfying (C.3) is

$$\bar{T} = -\frac{g}{R}\frac{\Delta z}{\Delta P} \tag{C.4}$$

$$\bar{\pi} = \frac{\Delta\pi}{K\Delta P} \qquad P \equiv \ln p \tag{C.5}$$

hydrostatic equation for partial layer

$$z = z_o - \frac{R}{g}(\bar{T} - b\bar{\pi})(P - P_o) - \frac{c_p b}{g}(\pi - \pi_o) \tag{C.6}$$

pressure at a given height

The equation must be solved numerically. One method follows. Let

$$p' = p_1 \exp\left[-\frac{g}{R\bar{T}}\delta z\right] \tag{C.7}$$

$$\delta\pi^n = \left(\frac{p^n}{p_{oo}}\right)^K - \left(\frac{p_o}{p_{oo}}\right)^K, \quad p_{oo} = 1000\,mb \tag{C.8}$$

$$\delta P^{n+1} = -\frac{g}{R(\bar{T} - b\bar{\pi})}\left[\delta z + \frac{c_p b}{g}\delta\pi^n\right] \tag{C.9}$$

$$P^{n+1} = \delta P^{n+1} + P_o \qquad (C.10)$$

$$p^{n+1} = exp(P^{n+1}) \qquad (C.11)$$

For b=10 deg the scheme converges to 9 decimal places in about 5 scans.

Appendix D. $T \propto Z$

temperature law

$$T = \overline{T} + b(z - \overline{z}) \tag{D.1}$$

hydrostatic equation for layer (b≠0)

$$\Delta z = \overline{z} - z + \frac{1}{b}\left\{\left[\overline{T} + b(z - \overline{z})\right]\left(\frac{p + \Delta p}{p}\right)^{-\frac{Rb}{g}} - \overline{T}\right\} \tag{D.2}$$

$\overline{T}, \overline{p}$ equation (b≠0)

Solving (D.2) for $\overline{T}$ gives

$$\overline{T} = -b\left(z - \overline{z} + \frac{\Delta z}{1 - \alpha}\right) \tag{D.3}$$

where

$$\alpha = \left(\frac{p + \Delta p}{p}\right)^{-\frac{Rb}{g}} \tag{D.4}$$

choice for $\overline{T}, \overline{p}$ (b≠0)

One natural choice is

$$\overline{z} = z + \frac{\Delta z}{2} \tag{D.5}$$

and then

$$\overline{T} = -\frac{b \Delta z}{2}\left(\frac{1 + \alpha}{1 - \alpha}\right) \tag{D.6}$$

IF b=0, then (D.2) becomes

$$\Delta z = -\frac{R}{g}\,\overline{T}\,\Delta P \tag{D.2a}$$

giving

$$\overline{T} = -\frac{g}{R}\,\frac{\Delta z}{\Delta P} \tag{D.3a}$$

and the choice of $\overline{Z}$ is open. Certainly, (D.5) can still beused.

hydrostatic equation for partial layer

b≠0

$$z = z_0 + \left(\frac{\alpha - 1}{b}\right)\left[\overline{T} + b(z_0 - \overline{z})\right] \tag{D.7}$$

b=0

$$z = z_o - \frac{R}{g} \overline{T} (P - P_o) \tag{D.7}$$

pressure at a given height

b≠0

$$p = p_o \frac{\overline{T} + b(z - \overline{z})}{\overline{T} - \frac{b}{2} \Delta z} \tag{D.8}$$

b=0

$$P = P_o - \frac{g}{R\overline{T}} (z - z_o), \quad p = \exp(P) \tag{D.8a}$$

Probably (D.8a) should be used for a range of b near zero.

Appendix E. $\theta \propto \ln p$

temperature law

$$\theta = \tilde{\theta} + b(P - \tilde{P}) \tag{E.1}$$

hydrostatic equation for layer

$$\Delta z = -\frac{c_p}{g}\frac{b}{K}\left\{\pi\left[\ln(\pi+\Delta\pi) - \ln\pi\right] + \Delta\pi\left[\ln(\pi+\Delta\pi) - 1\right]\right\} \tag{E.2}$$

$\tilde{\theta}, \tilde{P}$ equation

Solving (E.2) for $\tilde{\theta}$ gives

$$\tilde{\theta} = -\frac{g}{c_p}\frac{\Delta z}{\Delta\pi} - b\left\{P_o - \tilde{P} + \frac{1}{K}\left[\frac{\pi}{\Delta\pi}\ln\left(\frac{\pi+\Delta\pi}{\pi}\right) + \ln\left(\frac{\pi+\Delta\pi}{e}\right)\right]\right\} \tag{E.3}$$

choice of $\tilde{\theta}, \tilde{\pi}$

One natural choice is

$$\tilde{\theta} = -\frac{g}{c_p}\frac{\Delta z}{\Delta\pi} \tag{E.4}$$

and thus

$$\tilde{P} = P_o + \frac{\pi}{K\Delta\pi}\ln\left(\frac{\pi+\Delta\pi}{\pi}\right) + \frac{1}{K}\ln\left(\frac{\pi+\Delta\pi}{e}\right) \tag{E.5}$$

hydrostatic equation for partial layer

$$z = z_o - \frac{c_p}{g}\frac{b}{K}\left[\pi_o \ln\left(\frac{\pi}{\pi_o}\right) + (\pi - \pi_o)\ln\left(\frac{\pi}{e}\right)\right] \tag{E.6}$$

pressure at a given height

An interative solution is given below. Begin with

$$\pi' = \pi_o - \frac{g}{c_p\tilde{\theta}}(z - z_o) \tag{E.7}$$

Then use

$$\pi^{n+1} = \pi_o - \frac{\frac{g}{c_p}(z - z_o) + \frac{b}{K}\left[\pi_o \ln\left(\frac{\pi^n}{\pi_o}\right) + (\pi^n - \pi_o) \ln\left(\frac{\pi^n}{e}\right)\right]}{\bar{\theta} + b(P_o - \bar{P})} \qquad \text{(E.8)}$$

Iteration with b=30 deg gives 9-place accuracy after about 6 scans.

Appendix F. $\theta \propto p$

temperature law

$$\theta = \bar{\theta} + b(p - \bar{p}) \qquad \text{(F.1)}$$

hydrostatic equation for layer

$$\Delta z = -\frac{c_p}{g}(\bar{\theta} - b\bar{p})\Delta\pi - \frac{c_p b p_{oo}}{g}\left(\frac{K}{K+1}\right)\left[(\pi+\Delta\pi)^{\frac{K+1}{K}} - \pi^{\frac{K+1}{K}}\right] \qquad \text{(F.2)}$$

$\bar{\theta}$, $\bar{P}$ equation

Solving (F.2) for $\bar{\theta}$ gives

$$\bar{\theta} = -\frac{g}{c_p}\frac{\Delta z}{\Delta\pi} + b\left\{\bar{p} - \frac{p_{oo}}{\Delta\pi}\left(\frac{K}{K+1}\right)\left[(\pi+\Delta\pi)^{\frac{K+1}{K}} - \pi^{\frac{K+1}{K}}\right]\right\} \qquad \text{(F.3)}$$

choice of $\bar{\theta}, \bar{p}$

A natural choice is

$$\bar{\theta} = -\frac{g}{c_p}\frac{\Delta z}{\Delta\pi} \qquad \text{(F.4)}$$

$$\bar{p} = \frac{p_{oo}}{\Delta\pi}\left(\frac{K}{K+1}\right)\left[(\pi+\Delta\pi)^{\frac{K+1}{K}} - \pi^{\frac{K+1}{K}}\right] \qquad \text{(F.5)}$$

hydrostatic equation for partial layer

$$z = z_o - \frac{c_p}{g}(\bar{\theta} - b\bar{p})(\pi - \pi_o) - \frac{c_p b p_{oo}}{g}\left(\frac{K}{K+1}\right)\left(\pi^{\frac{K+1}{K}} - \pi_o^{\frac{K+1}{K}}\right) \qquad \text{(F.6)}$$

pressure at a given height

A numerical solution is needed. One possibility follows.

$$\pi' = \pi_o - \frac{g}{c_p\bar{\theta}}(z - z_o) \qquad \text{(F.7)}$$

$$\pi^{n+1} = \pi_o - \frac{\frac{g}{c_p}(z - z_o) + \left(\frac{K}{K+1}\right) b p_{oo}\left[(\pi^n)^{\frac{K}{K+1}} - \pi_o^{\frac{K}{K+1}}\right]}{\bar{\theta} - b\bar{p}} \qquad \text{(F.8)}$$

With b=.06 deg/mb this scheme takes about 13 scanes to converge to 9-place accuracy.

Appendix G. $\theta \propto \pi$

temperature law

$$\theta = \bar{\theta} + b(\pi - \bar{\pi}) \tag{G.1}$$

hydrostatic equation for layer

$$\Delta z = -\frac{c_p}{g}(\bar{\theta} - b\bar{\pi})\Delta\pi - \frac{c_p}{g} b\left(\pi + \frac{\Delta\pi}{2}\right)\Delta\pi \tag{G.2}$$

$\bar{\theta}, \bar{\pi}$ equation

Solving (G.2) for $\bar{\theta}$ gives

$$\bar{\theta} = -\frac{g}{c_p}\frac{\Delta z}{\Delta\pi} + b\left[\bar{\pi} - \left(\pi + \frac{\Delta\pi}{2}\right)\right] \tag{G.3}$$

choice of $\bar{\theta}, \bar{\pi}$

A natural choice is

$$\bar{\theta} = -\frac{g}{c_p}\frac{\Delta z}{\Delta\pi} \tag{G.4}$$

$$\bar{\pi} = \pi + \frac{\Delta\pi}{2} \tag{G.5}$$

hydrostatic equation for partial layer

$$z = z_0 - \frac{c_p}{g}\left[(\bar{\theta} - b\bar{\pi}) + b\left(\frac{\pi + \pi_0}{2}\right)\right](\pi - \pi_0) \tag{G.6}$$

pressure at a given height

$$\delta\pi = \frac{-\frac{2g}{c_p}\,\delta z}{\left[\bar{\theta} - b(\bar{\pi} - \pi_0)\right] + \left\{\left[\bar{\theta} - b(\bar{\pi} - \pi_0)\right]^2 + \frac{2gb}{c_p}\,\delta z\right\}^{1/2}} \tag{G.7}$$

Appendix H. $\theta \propto z$

temperature law

$$\theta = \bar{\theta} + b(z - \bar{z}) \tag{H.1}$$

hydrostatic equation for layer

$$\Delta z = \bar{z} - z + \frac{1}{b}\left\{\beta\left[\bar{\theta} + b(z - \bar{z})\right] - \bar{\theta}\right\} \tag{H.2}$$

where

$$\beta = \exp\left(-\frac{Rb}{g}\Delta\pi\right) \tag{H.3}$$

$\bar{\theta}$, $\bar{z}$ equation

(H.2) $\Rightarrow$

$$\bar{\theta} = b\left(\bar{z} - z - \frac{\Delta z}{1-\beta}\right) \tag{H.4}$$

choice of $\bar{\theta}$, $\bar{z}$

One reasonable choice is

$$\bar{z} = z + \frac{\Delta z}{2} \tag{H.5}$$

giving

$$\bar{\theta} = -\frac{b\Delta z}{2}\left(\frac{1+\beta}{1-\beta}\right) \tag{H.6}$$

hydrostatic equation for partial layer

$$z = z_o - \frac{c_p}{g}\left[(\bar{\theta} - b\bar{\pi}) + b\left(\frac{\pi + \pi_o}{2}\right)\right](\pi - \pi_o) \tag{H.7}$$

pressure at a given height

$$\pi = \pi_o - \frac{g}{Rb}\ln\left[\frac{\bar{\theta} + b(z - \bar{z})}{\bar{\theta} + b(z_o - \bar{z})}\right] \tag{H.8}$$

$$p = p_{oo}\,\pi^{1/\kappa} \tag{H.9}$$

Figures

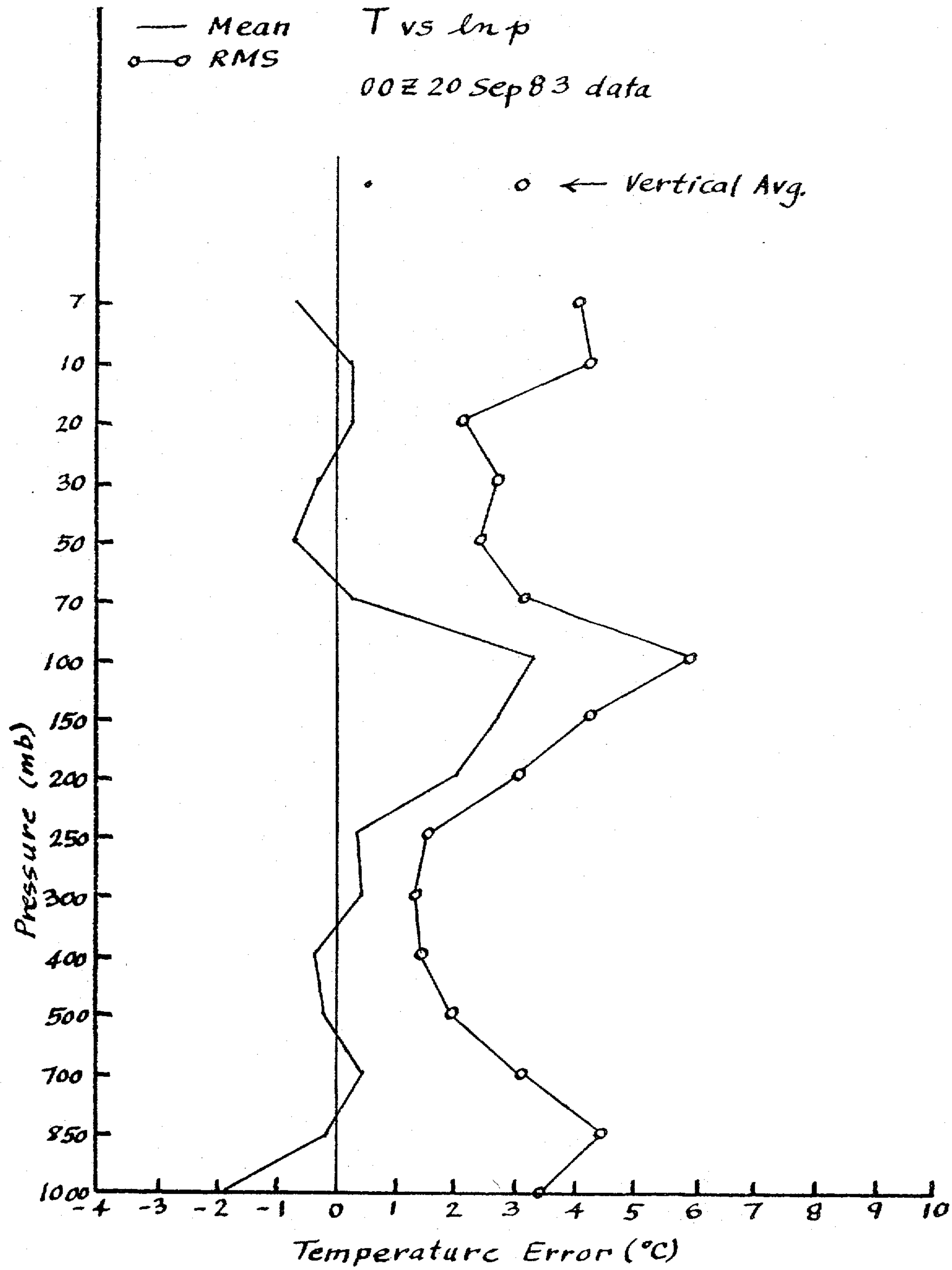
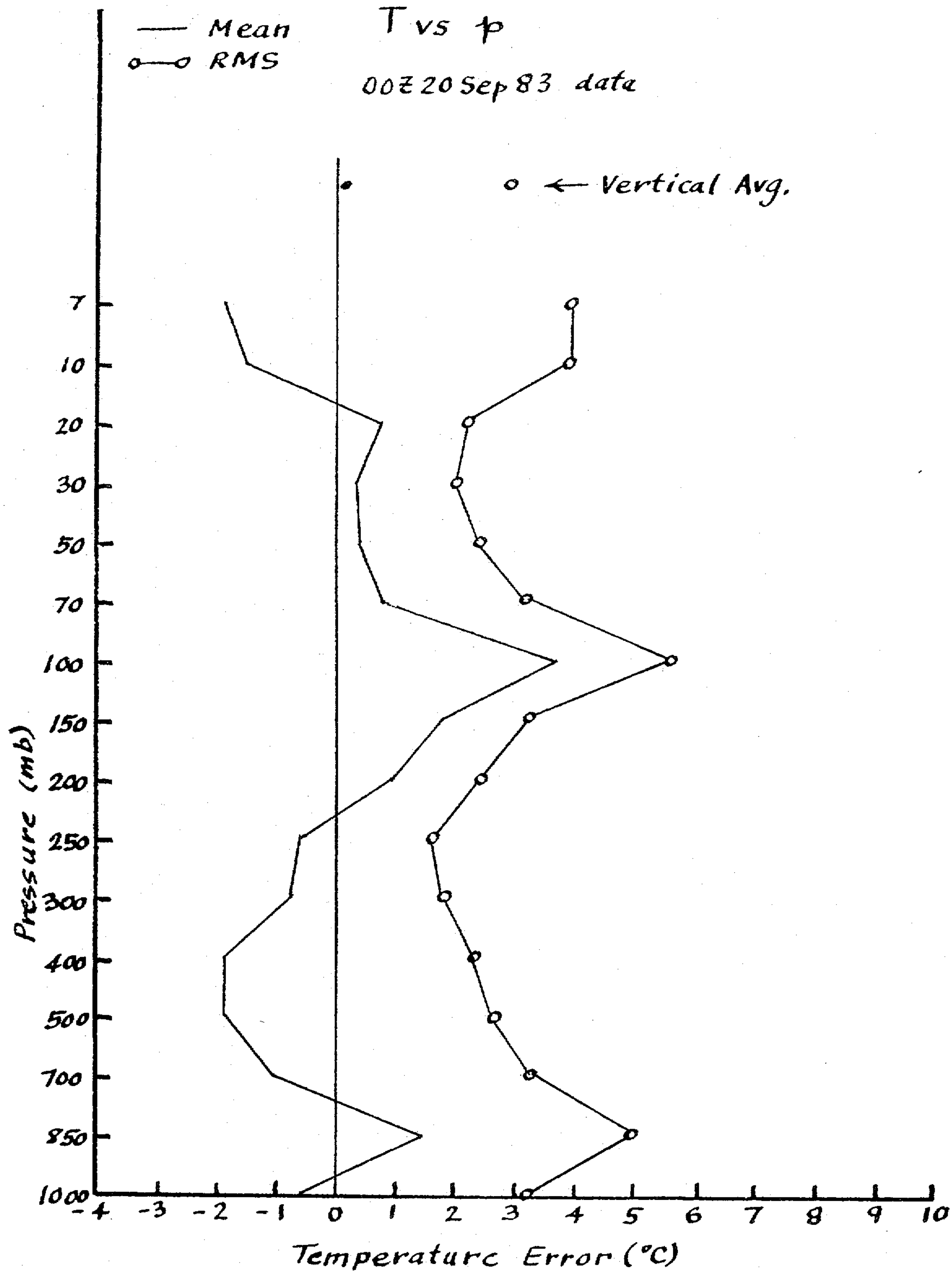

Mean
RMS
T vs ln p
00Z 20 Sep 83 data
Vertical Avg.
7
10
20
30
50
70
100
150
200
250
300
400
500
700
850
1000
Pressure (mb)
-4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
Temperature Error (°C)

Fig. 2



Mean
RMS
T vs p
00Z 20 Sep 83 data
← Vertical Avg.
Pressure (mb)
7
10
20
30
50
70
100
150
200
250
300
400
500
700
850
1000
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
Temperature Error (°C)

Fig. 3

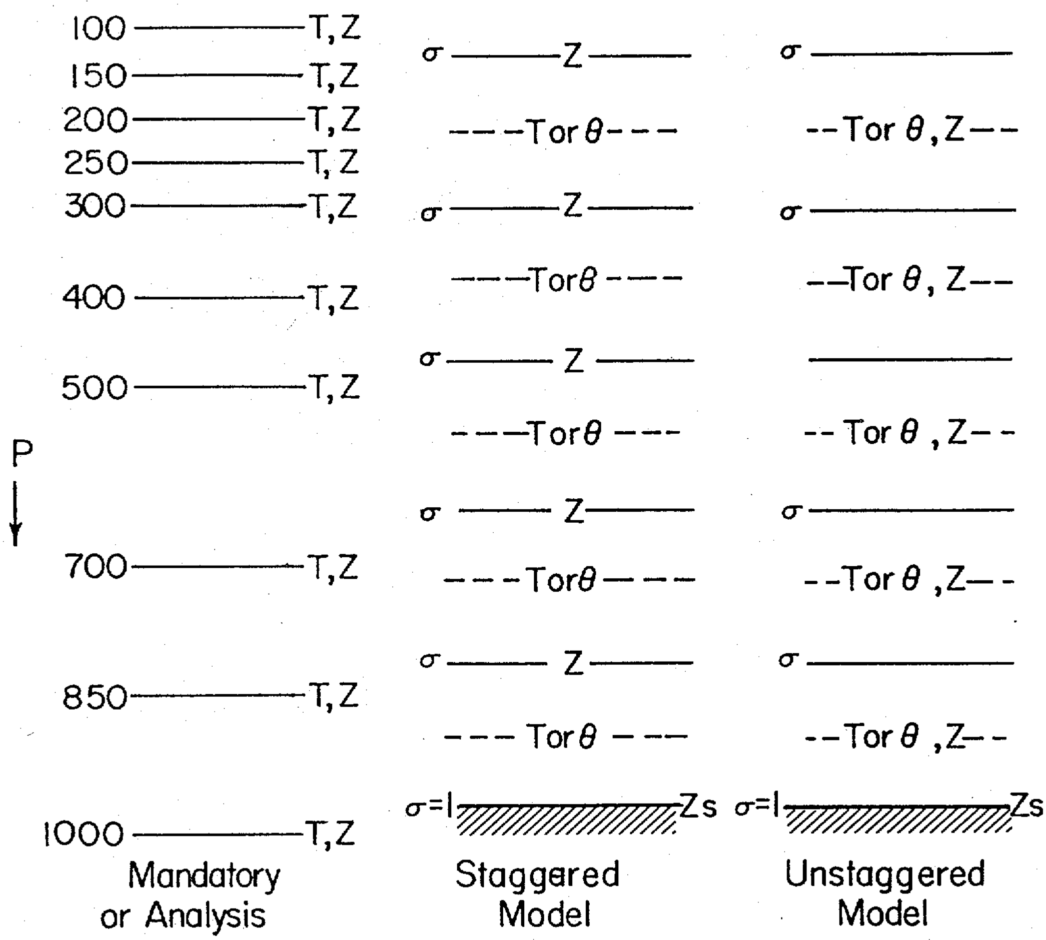
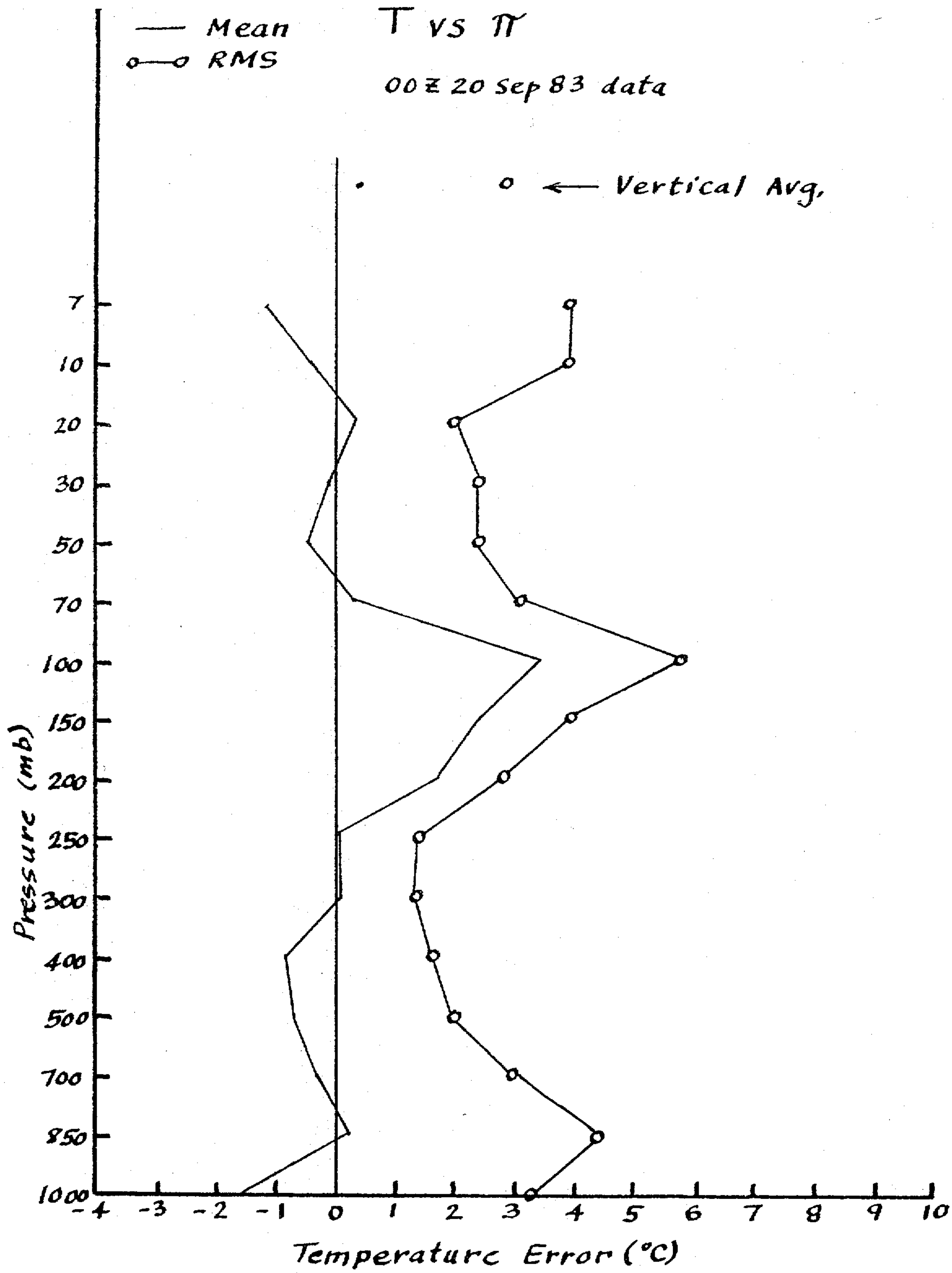
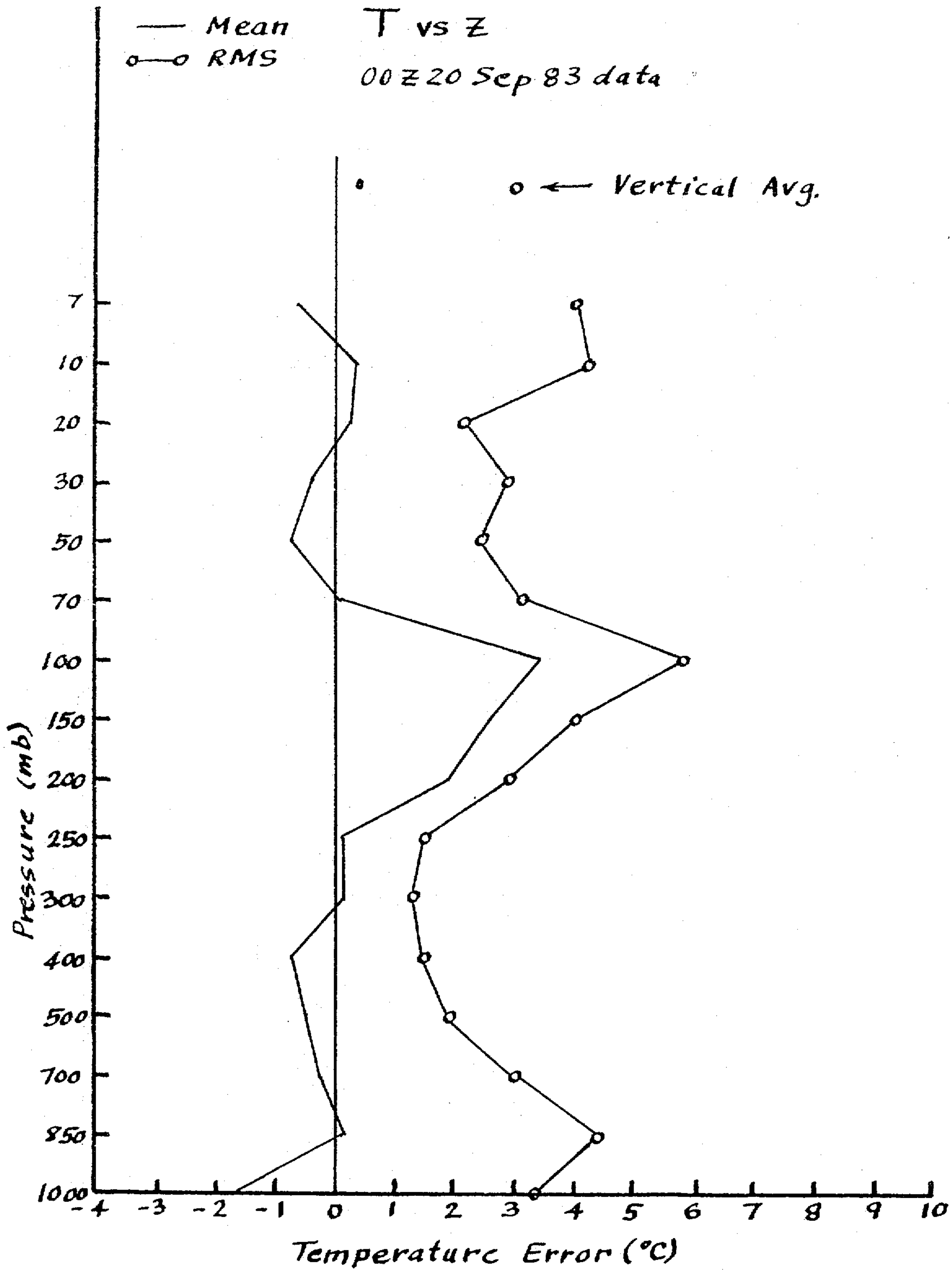


Fig 12 Data arrangement



Mean
RMS
T vs π
00Z 20 Sep 83 data
Vertical Avg.
7
10
20
30
50
70
100
150
200
250
300
400
500
700
850
1000
Pressure (mb)
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
Temperature Error (°C)

Fig. 4



Mean
RMS
T vs Z
00Z 20 Sep 83 data
Vertical Avg.
Pressure (mb)
7
10
20
30
50
70
100
150
200
250
300
400
500
700
850
1000
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
Temperature Error (°C)

Fig. 5

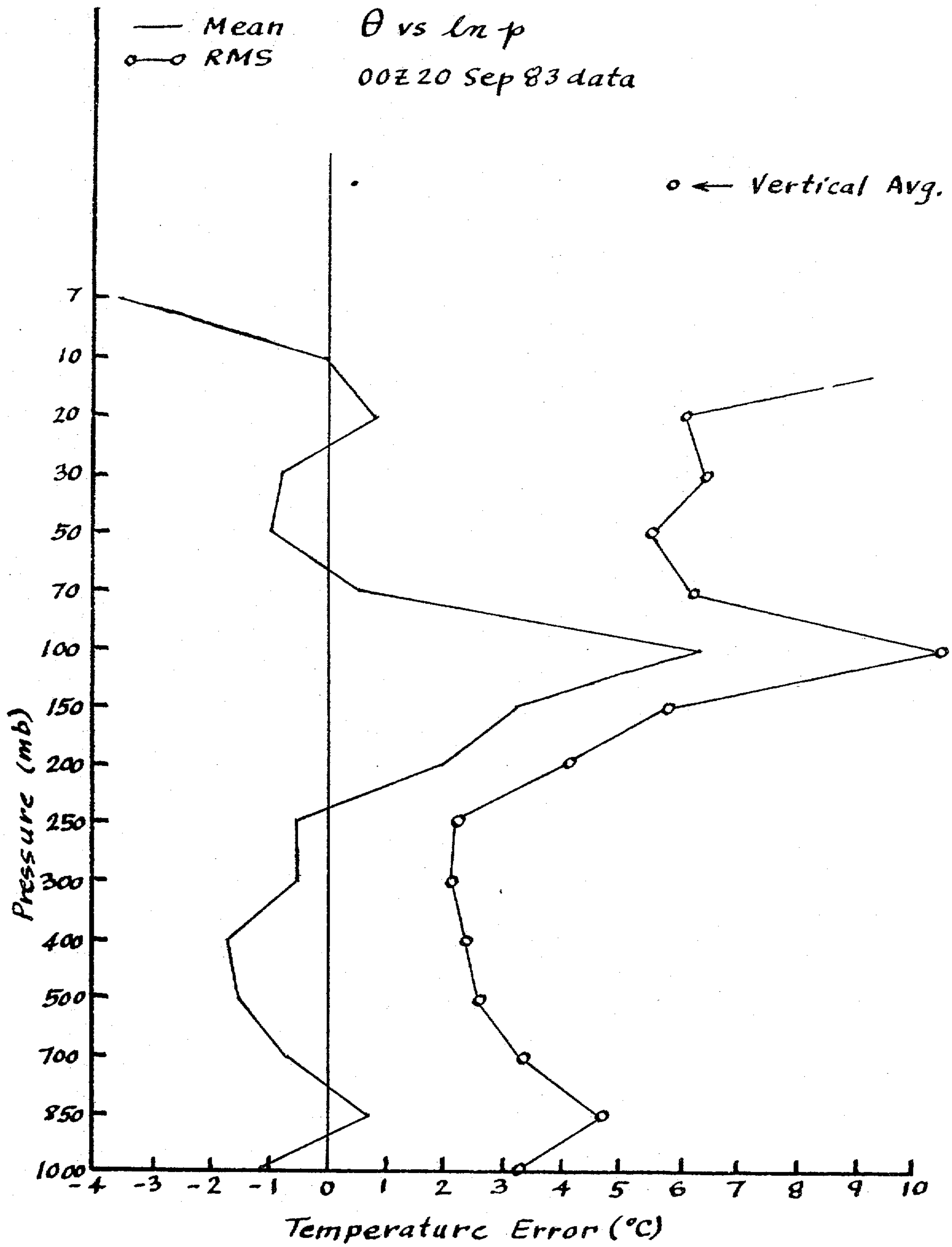

— Mean
o—o RMS
θ vs ln p
00Z 20 Sep 83 data
o ← Vertical Avg.
7
10
20
30
50
70
100
150
200
250
300
400
500
700
850
1000
Pressure (mb)
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8
9
10
Temperature Error (°C)

Fig 6

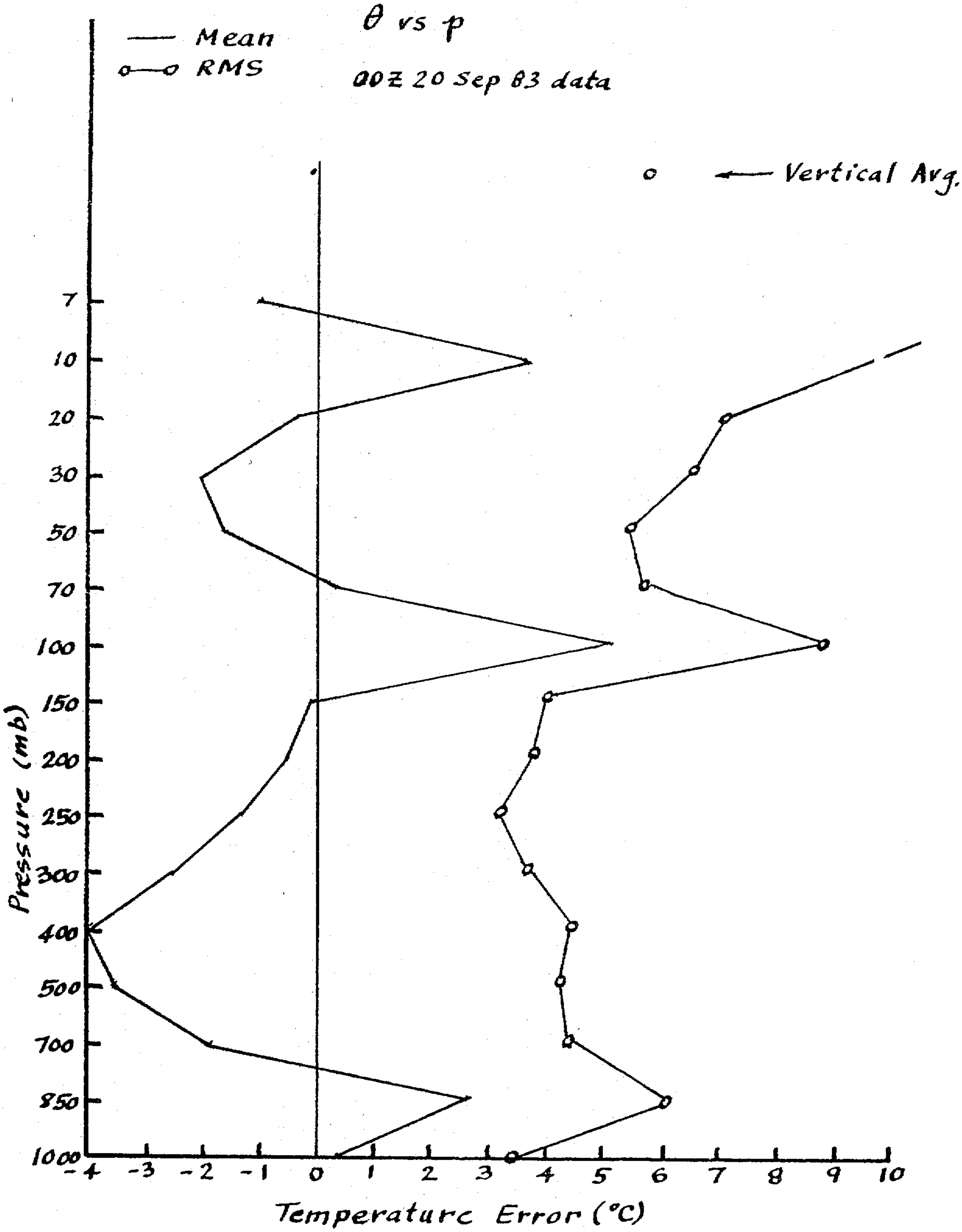

Mean
RMS
θ vs p
00Z 20 Sep 83 data
Vertical Avg.
7
10
20
30
50
70
100
150
200
250
300
400
500
700
850
1000
Pressure (mb)
-4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
Temperature Error (°C)

Fig. 7

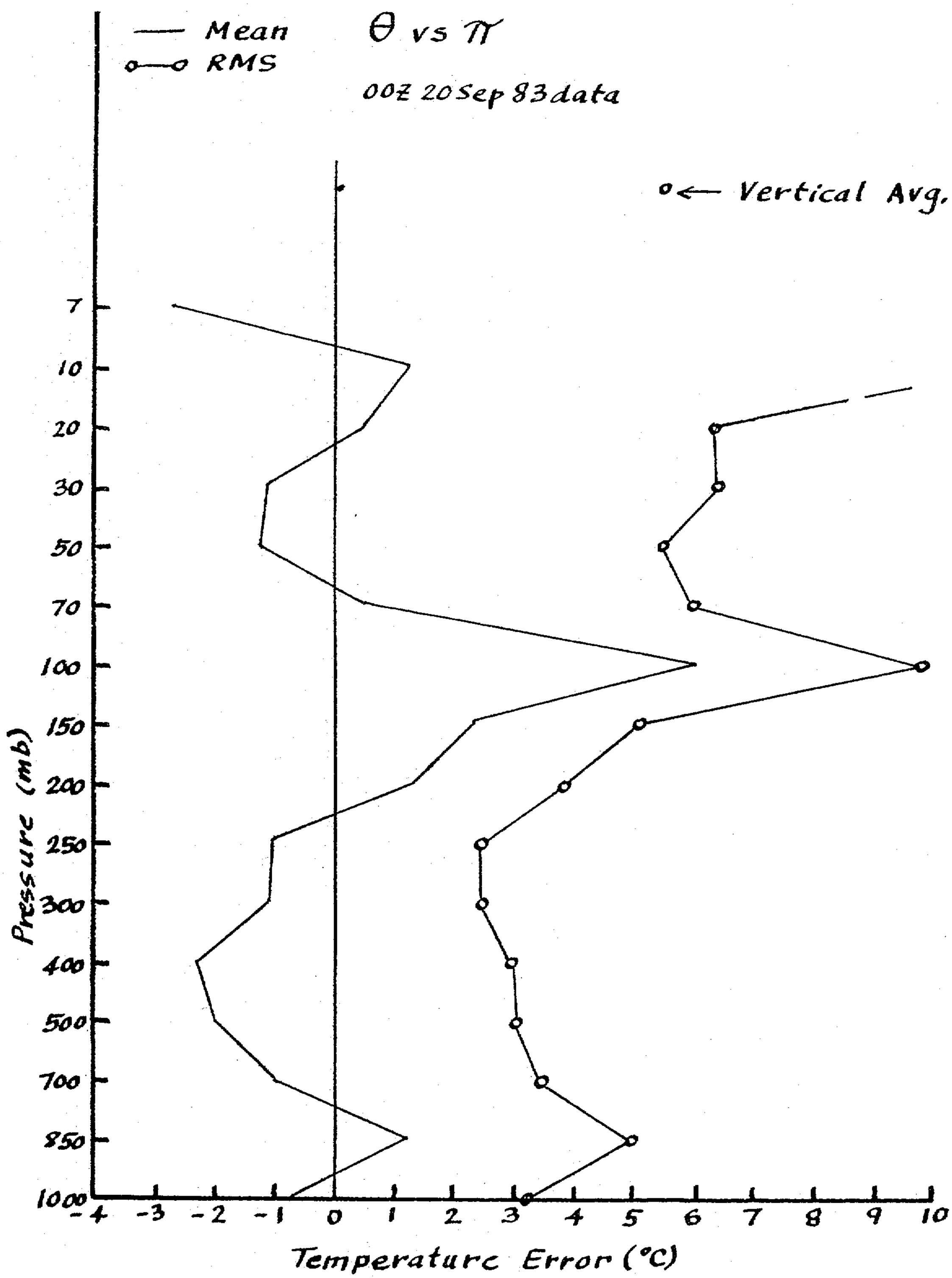

Mean
RMS
θ vs π
00Z 20Sep 83data
Vertical Avg.
7
10
20
30
50
70
100
150
200
250
300
400
500
700
850
1000
Pressure (mb)
-4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
Temperature Error (°C)

Fig. 8

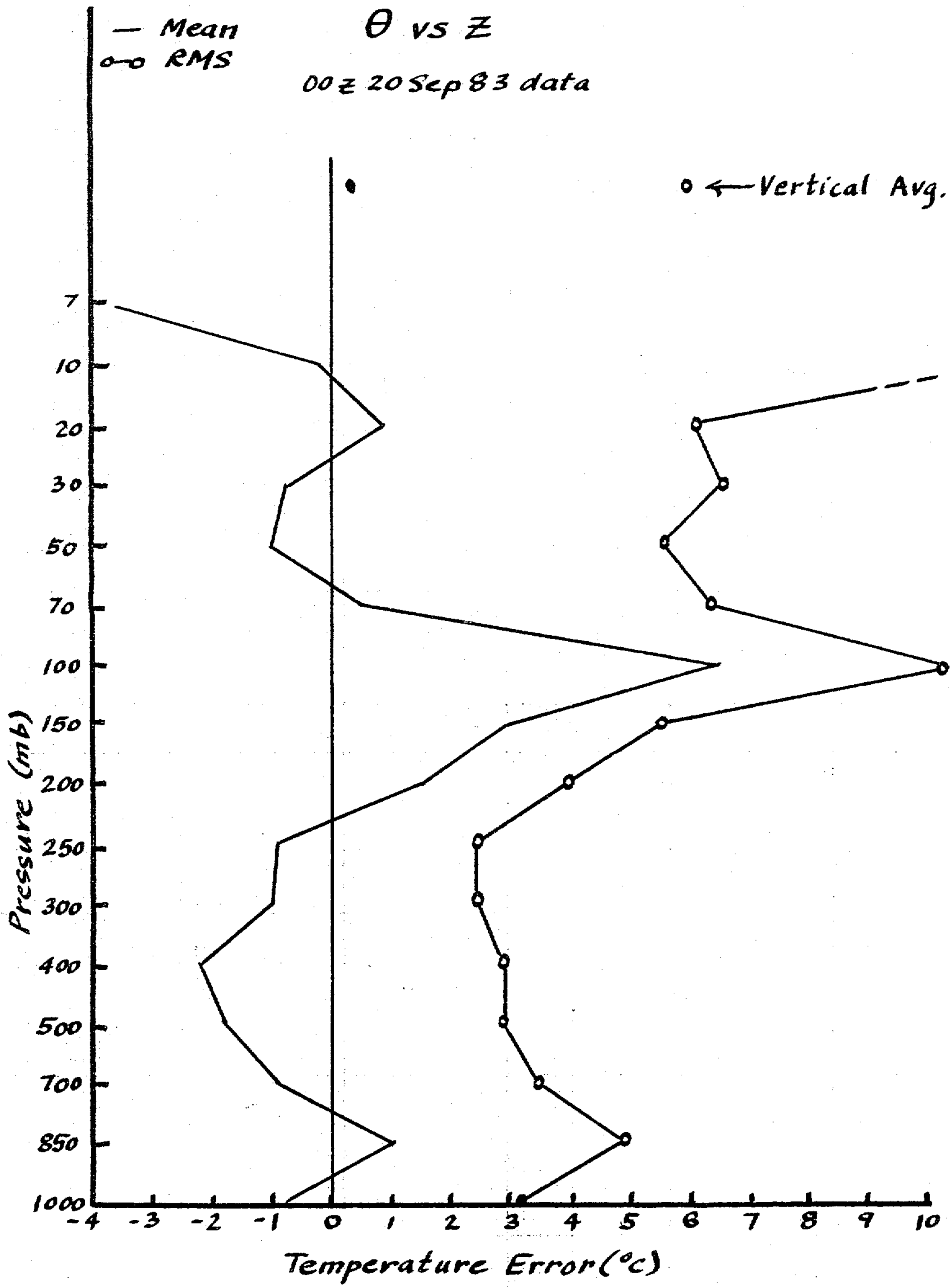
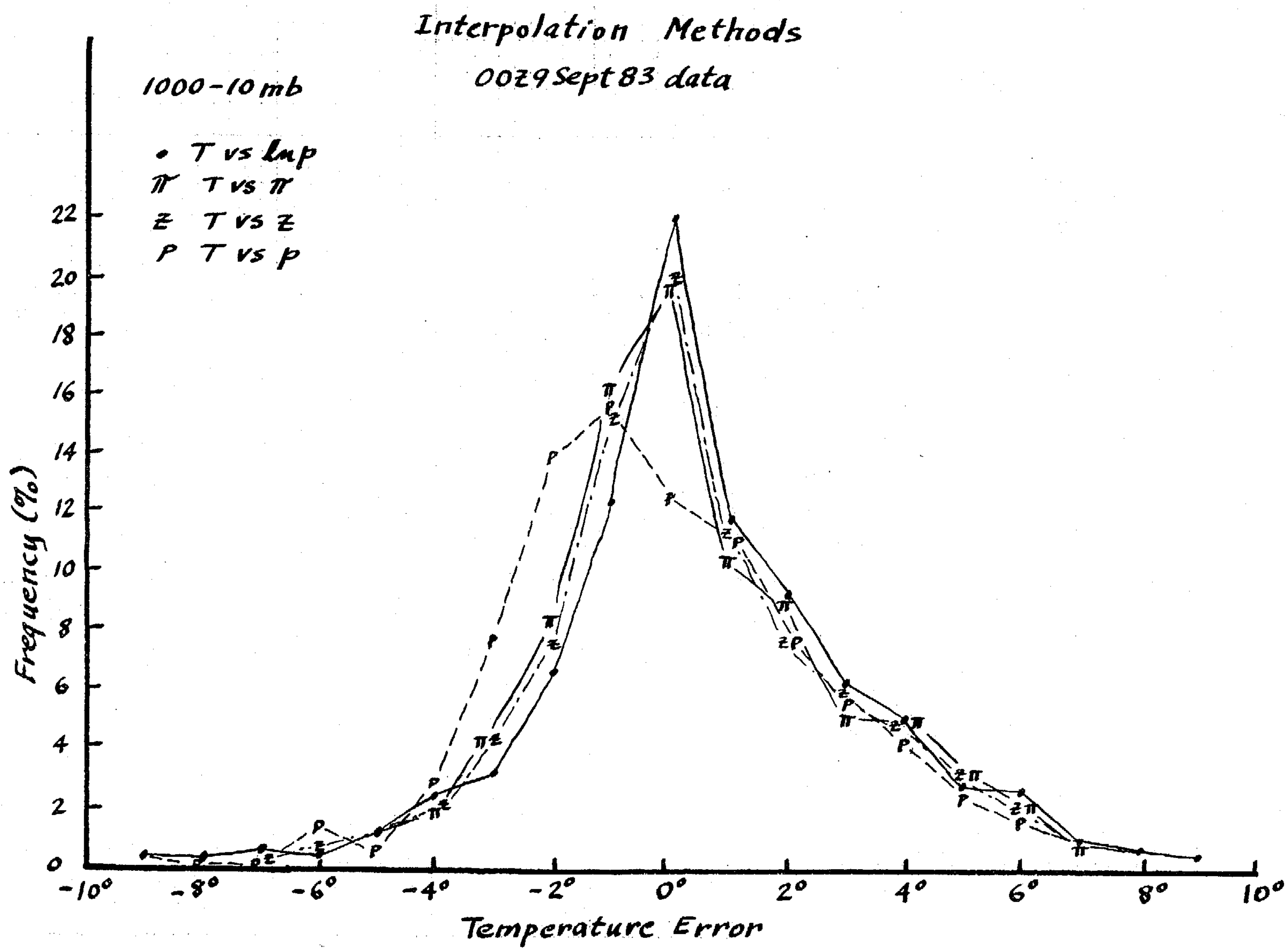
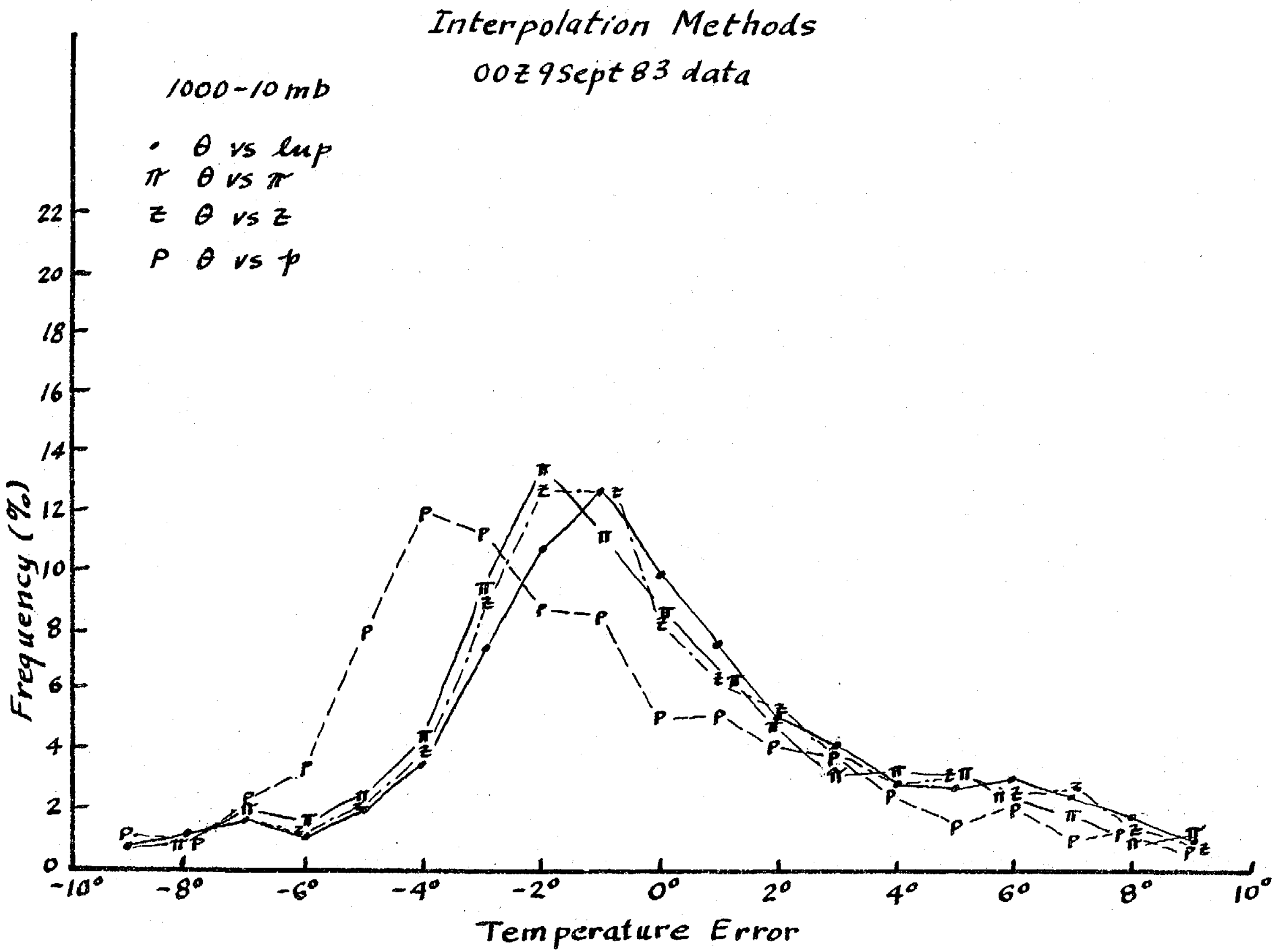

— Mean
o-o RMS
θ vs Z
00Z 20 Sep 83 data
o ← Vertical Avg.
Pressure (mb)
7
10
20
30
50
70
100
150
200
250
300
400
500
700
850
1000
-4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10
Temperature Error (°C)

Fig. 9.



Interpolation Methods
0029 Sept 83 data
1000-10 mb
• T vs lnp
π T vs π
z T vs z
p T vs p
Frequency (%)
0
2
4
6
8
10
12
14
16
18
20
22
-10°
-8°
-6°
-4°
-2°
0°
2°
4°
6°
8°
10°
Temperature Error

Fig. 10.



Interpolation Methods
00Z 9 Sept 83 data
1000-10 mb
• θ vs lnp
π θ vs π
z θ vs z
p θ vs p
Frequency (%)
0
2
4
6
8
10
12
14
16
18
20
22
-10°
-8°
-6°
-4°
-2°
0°
2°
4°
6°
8°
10°
Temperature Error

Fig. 11

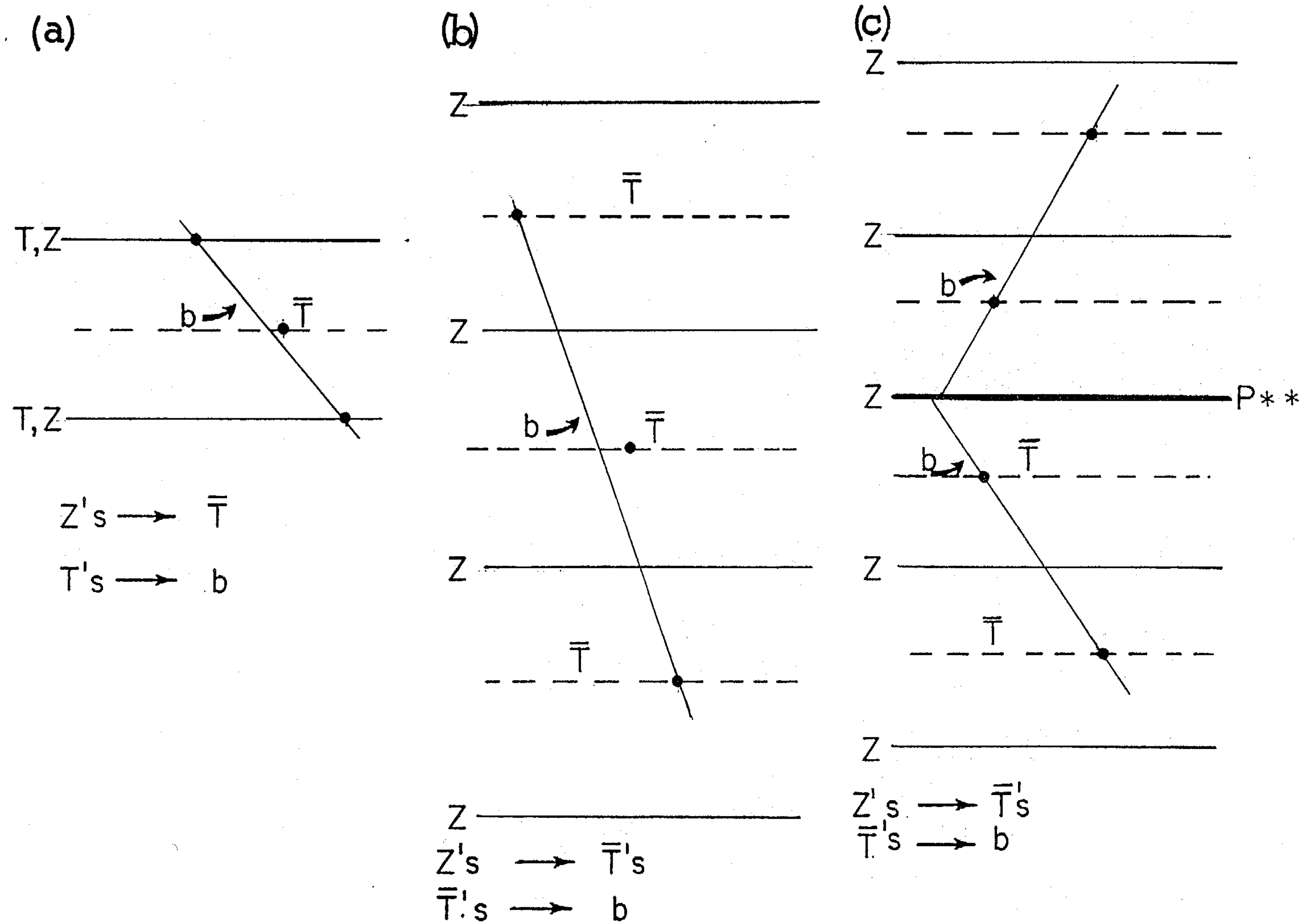


Fig. 13. Data arrangement for determination of lapse rates

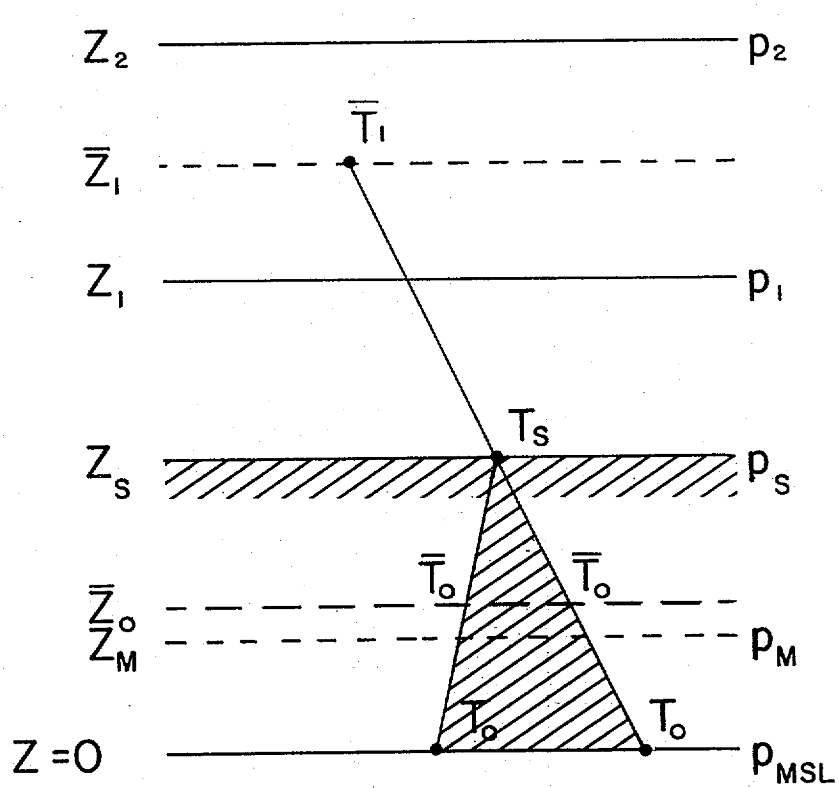


Fig 14 Data arrangement for SHUELL pressure reduction

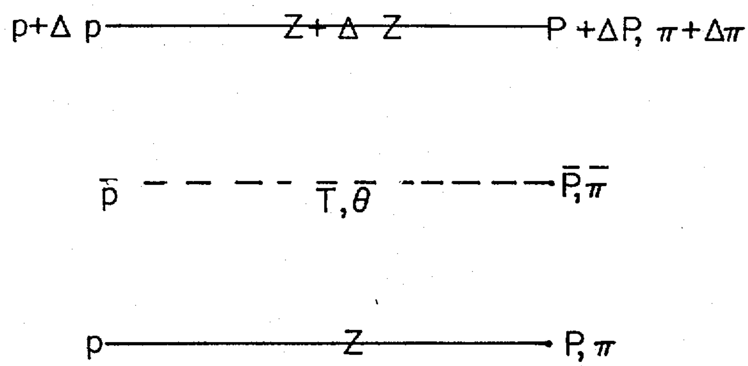


Fig. 15 Data structure for layer.

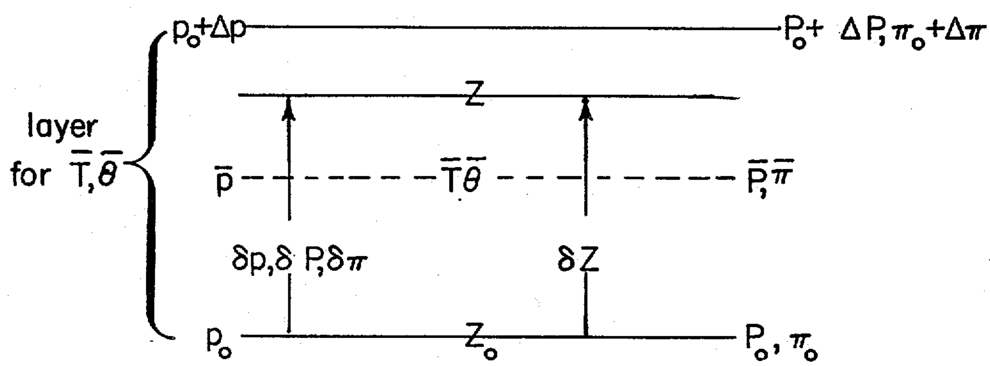


Fig. 16 Data structure for partial layer.

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