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DERIVATION AND SUGGESTED METHOD OF THE APPLICATION OF SIMPLIFIED RELATIONS FOR SURFACE FLUXES IN THE MEDIUM-RANGE FORECAST MODEL: UNSTABLE CASE

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I. Introduction

An economical technique is presented for solving the boundary layer equations that yield the surface fluxes of momentum, heat, and moisture for unstable conditions. The method is intended to replace the iterative scheme used in the National Meteorological Center's Medium Range Forecast model (MRF).

The MRF formulation uses Obukhov similarity theory which posits the existence of nondimensional 'universal' functions (φ_M , φ_H , φ_Q) that are functions of only Z/L, where L is the Obukhov length. The $\varphi_{M,H,Q}$ functions give the nondimensional vertical shears of wind speed, potential temperature, and specific humidity:

$$\frac{kz}{u_*}\frac{\partial U}{\partial z} = \varphi_M(z/L) \tag{1.1}$$

$$\frac{kz}{\theta_*}\frac{\partial \theta}{\partial z} = \varphi_H(z/L) \tag{1.2}$$

$$\frac{kz}{q_*}\frac{\partial q}{\partial z} = \varphi_Q(z/L) . \tag{1.3}$$

The use of the von Karman constant k ($\doteq 0.4$) in (1.1, 2, 3) is merely historical and is not required by the Obukhov similarity theory. The turbulent scaling parameters u *, θ*, q* have the same units as wind speed, potential temperature, and specific humidity and define the surface turbulent fluxes. That is,

$$u_* \equiv \sqrt{|\vec{\tau}|/\rho} = \sqrt{-F_M/\rho} \tag{1.4}$$

$$\theta_* \equiv -F_H/\rho C_p u_* \tag{1.5}$$

$$q_* \equiv -F_Q/\rho \mathcal{L} u_* . \tag{1.6}$$

In these relations, $F_{M,H,Q}$ are the (surface) fluxes of momentum, heat, and humidity; τ is the surface stress; ρ is the density of air; C_p is the specific heat of air at constant pressure; $\mathcal{L}$ is the latent heat of evaporation. Rewriting (1.1-6), we have,

$$\mathcal{F}_M = -\rho u_*^2 = -\rho (kz)^2 \varphi_M^{-2} \frac{\partial U}{\partial z} \tag{1.7}$$

$$\mathcal{F}_H = -\rho c_p u_* \theta_* = -\rho c_p (kz)^2 (\varphi_M \varphi_H)^{-1} \frac{\partial U}{\partial z} \frac{\partial \theta}{\partial z} \tag{1.8}$$

$$\mathcal{F}_Q = -\rho \mathcal{L} u_* q_* = -\rho \mathcal{L} (kz)^2 (\varphi_M \varphi_Q)^{-1} \frac{\partial U}{\partial z} \frac{\partial q}{\partial z} . \tag{1.9}$$

Many flux-profile relations have been proposed for $\varphi_{M,H,Q}$ (see, for example, Yaglom, 1977, for a survey). Most of the relations for the unstable case that have been used in numerical forecast or simulation models are of the form

$$\varphi_M(\zeta) = a_M (1 - \alpha_M \zeta)^{-\beta_M}, \tag{1.10}$$

$$\varphi_Q(\zeta) = \varphi_H(\zeta) = a_H(1 - \alpha_H \zeta)^{-\beta_H}. \tag{1.11}$$

The Businger et al. (1971) formulation is probably the most frequently used formulation in numerical forecast modeling. For the Businger relations, the relevant constants are: $a_M = 1, a_H = 0.74, \alpha_M = 15, \alpha_H = 9, \beta_H^2 = \beta_M = 1/4$. The relations proposed earlier by Dyer (1967) use a $a_{M,H} = 1$, $\alpha_{M,H} = 15, \beta_M = 0.275$ and $\beta_H = 0.55$. The MRF physics formulation uses the $a_{M,H} = 1, \alpha_{M,H} = 16, \beta_M = \beta_H^2 = 1/4$ proposed later by Dyer (1974) and Hicks (1976):

$$\varphi_M(\zeta) = (1 - 16\zeta)^{-1/4} \tag{1.12}$$

$$\varphi_H(\zeta) = (1 - 16\zeta)^{-1/2} = \varphi_Q(\zeta). \tag{1.13}$$

From a numerical viewpoint, Dyer's original relations are cumbersome since they result in no known closed-form integrals of (2.14 - 17). These integrals are needed for the iterative solution for ζ that is required for the computation of $F_{M,H,Q}$ (see section II). On the otherhand, the selection of the Dyer-Hicks relations for the surface layer turbulent formulation is particularly fortunate, since the equality of $\alpha_M = \alpha_H$, $\varphi_H^2 = \varphi_M$, $\beta_M = \beta_H^2$ and $a_M = a_H = 1$ simplifies the solution for ζ compared to either Businger's or to Dyer's earlier formulation.

It must be recognized, however, that the precise values of $\alpha_{M,H}$, $\beta_{M,H}$ and $a_{M,H}$ are not known and are subject to controversy. For example, Dyer (1974), in a review of several flux-profile relationships, comments: "the results of Businger <u>et al.</u> (1971) remain a difficulty which calls for considerable clarification". In addition, Carl <u>et al.</u> used a composite of tower data to determine $\varphi_{M,H}(\zeta)$. Their results are,

$$\varphi_M(\zeta) \doteq (1 - 16\zeta)^{-1/3} \tag{1.14}$$

$$\varphi_H(\zeta) \doteq 0.74(1 - 16\zeta)^{-1/4} \tag{1.15}$$

and also

The Businger and Dyer-Hicks flux-profile laws stand in sharp contrast to the free-convection profile laws: the free-convection relations require that the turbulent fluxes become independent of wind shear for large $-\xi$. This distinction is important since computational experience shows that the Businger-Dyer-Hicks solutions for ξ, Φ_M and Φ_H can run out of control for progressively smaller wind shears, $\partial U / \partial z \rightarrow 0$. We will return to this evidently important distinction in Section II, but for now we shall assume that (1.12 - 13) are correct for domains in which we shall apply them.

II. Solutions of the Flux-Profile Relations

A. General

In this section we develop some relations between $\xi \, (= z/L)$, the gradient Richardson number (R), and the bulk Richardson number (R_B). The bulk Richardson number is used to determine ξ. The surface fluxes can then be computed from ξ.

We begin by substituting (1.1, 2) into a defining relation for the Obukhov length,

$$L = \frac{\bar{\theta}}{kg} u_*^2 \, \theta_*^{-1} . \tag{2.1}$$

$$K_M \sim 2.5 \left(\frac{g F_H}{\rho c_p \theta} \right)^{1/3} (kz)^{4/3} . \qquad (1.16)$$

Panofsky (1978) suggested that the limiting form of K_H is

$$K_H \sim 1.5 \left(\frac{g F_H}{\rho c_p \theta} \right)^{1/3} (kz)^{4/3} . \qquad (1.17)$$

Eqs. (1.14), (1.16), and (1.17) are consistent with local free-convection, as is the Wyngaard, et al. (1978) comment that

$$\varphi_H(\xi) = 0.23(-\xi)^{-1/3} \qquad (1.18)$$

fits the Kansas data (Wyngaard and Coté, 1971) about as well as the Businger profile in the domain $0.5 \leq -\xi \leq 2$.

The result is

$$\zeta = \frac{\varphi^2_M(\zeta)}{\varphi_H(\zeta)} R , \tag{2.2}$$

in which the gradient Richardson number R is defined by

$$R = \frac{g}{\theta} \frac{\partial \theta}{\partial z} \Big/ \left(\frac{\partial U}{\partial z}\right)^2 . \tag{2.3}$$

The gradient Richardson number combined with the Dyer-Hicks flux-profile relations yields the simple result: for $R, \zeta < 0$

$$\zeta = R . \tag{2.4}$$

This equality between R and ζ (for $\zeta \leq 0$) is, unfortunately, of little practical value in numerical forecast models, such as the MRF, that have limited vertical resolution. In these models the local gradient Richardson number cannot be computed with fidelity, and the bulk Richardson number is usually the best that can be done. The bulk Richardson number between two levels z_1 and z_2 is defined as

$$R_B = \frac{g}{\bar{\theta}} \frac{h \Delta\theta}{U^2} . \tag{2.5}$$

In (2.5), g is the acceleration due to gravity; $\bar{\theta}$ is a representative (or average) potential temperature between z_1 and z_2; $\Delta\theta = \theta(z_2) - \theta(z_1)$; $U = U(z_2) - U(z_1)$; $h = z_2 - z_1$. For modeling purposes, z_2 is usually the height of the first layer or the midpoint of the first layer. In either case we generally have $z_2 >> z_1 = z_0$; $h = z_2 - z_1 \doteq z_2 = z$.

Expressions for U and $\Delta\theta$ may be derived by combining (1.1, 2) and (1.4 -6),

$$U(z) = \frac{u_*}{k} \int_{z_0}^{z} \frac{dz'}{z'} \left(1 - \alpha \frac{z'}{L}\right)^{-1/4} \tag{2.6}$$

$$\Delta\theta = \frac{\theta_*}{k} \int_{z_0}^{z} \frac{dz'}{z'} \left(1 - \alpha \frac{z'}{L}\right)^{-1/2} . \tag{2.7}$$

The results of the integration of (2.6) and (2.7) are,

$$U(z) = \frac{u_*}{k} F_M\left(z/L\,;\, z_o/L\right) \tag{2.8}$$

$$\Delta\theta = \frac{\theta_*}{k} F_H\left(z/L\,;\, z_o/L\right) \tag{2.9}$$

for which we have,

$$F_M(z/L\,;\,z_o/L) = \ln\left[\frac{(R_z-1)(R_o+1)^2}{(R_z+1)(R_o-1)}\right] + \tan^{-1}R_z - \tan^{-1}R_o \tag{2.10}$$

$$F_H(z/L\,;\,z_o/L) = \ln\left[\frac{(Q_z-1)(Q_o+1)}{(Q_z+1)(Q_o-1)}\right] \tag{2.11}$$

$$R_z = \left(1 - \alpha\frac{z}{L}\right)^{1/4}\,;\quad R_o = \left(1 - \alpha\frac{z_o}{L}\right)^{1/4} \tag{2.12 a,b}$$

$$Q_z = \left(1 - \alpha \frac{z}{L}\right)^{1/2} \quad ; \quad Q_0 = \left(1 - \alpha \frac{z_0}{L}\right)^{1/2} . \qquad (2.13\ a,b)$$

It is customary to express (2.8 - 13) in a somewhat different but otherwise equivalent form,

$$U = \frac{u_*}{k}\left[\ln \frac{z}{z_0} - \Psi_M(z/L\,;\,z_0/L)\right]; \qquad (2.14)$$

$$\Delta\Theta = \frac{\Theta_*}{k}\left[\ln \frac{z}{z_0} - \Psi_H(z/L\,;\,z_0/L)\right]; \qquad (2.15)$$

$$\Psi_M = \int_{z_0}^{z} \frac{dz'}{z'}\left[1 - \varphi_M(z'/L)\right]; \qquad (2.16)$$

$$\Psi_H = \int_{z_0}^{z} \frac{dz'}{z'}\left[1 - \varphi_H(z'/L)\right]. \qquad (2.17)$$

The functions $\Psi_{M,H}$ represent the deviation from near-neutrality: for conditions close to neutral, we have

$$U(z) \doteq \frac{u_*}{k} \ln \frac{z}{z_0} \; ; \; \Psi_M \to 0 \; ;$$

$$\Delta\Theta \doteq \frac{\theta_*}{k} \ln \frac{z}{z_0} \; ; \; \Psi_H \to 0 \; ;$$

$$L \to -\infty .$$

The 'standard form' (2.14 - 7) for $\Delta\Theta$ and U are more convenient to use than $F_{M,H}$. For conditions sufficiently close to neutral, an attempt to calculate F_M or F_H can lead to computational failure. This liability is not shared by the standard forms. The computational failure is caused by R_0 and Q_0 approaching unity for near-neutral conditions and small z_0 . With limited precision, $R_0 - 1$ and $Q_0 - 1$ can be erroneously computed as negative numbers with very small absolute values. This causes the arguments of the logarithmic functions to become large negative numbers, and the computations halts.

The transformation of $F_{M,H}$ to the standard form can be accomplished by manipulating (2.10, 11). By using some simple algebra, we have

$$\frac{(R_z-1)(R_0+1)^2}{(R_z+1)(R_0^2-1)}$$

$$= \frac{(R_z-1)(R_0+1)^2(R_0^2+1)}{(R_z+1)(R_0^4-1)}$$

$$= \frac{(R_z^4-1)(R_0+1)^2(R_0^2+1)}{(R_z^2+1)(R_z+1)^2(R_0^4-1)}$$

$$= \frac{z}{z_0}\frac{(1+R_0)^2(1+R_0^2)}{(1+R_z)^2(1+R_z^2)} \qquad (2.18)$$

from which

$$\Psi_M = \ln\left[\frac{(1+R_z)^2(1+R_z^2)}{(1+R_0)^2(1+R_0^2)}\right] - 2\tan^{-1}R_0 \qquad (2.19)$$
$$+ 2\tan^{-1}R_z$$

follows. A similar restructuring of F_H yields

$$\Psi_H = 2 \ln\left(\frac{1 + Q_z}{1 + Q_0}\right). \tag{2.20}$$

Since the factors in $\Psi_{M,H}$ are all greater than unity, there is no danger of computational failure as is the case with $F_{M,H}$. The computation of z/L from U, $\Delta\theta$, z, and z_0 is more difficult than from $\partial U/\partial z$ and $\partial\theta/\partial z$. The relation involving the gradient Richardson number

$$\xi = \frac{\varphi_M^2(\xi)}{\varphi_H(\xi)} R$$

is replaced by

$$\xi = \frac{F_M^2}{F_H} R_B = \frac{\left(\ln \frac{z}{z_0} - \Psi_M\right)^2}{\left(\ln \frac{z}{z_0} - \Psi_H\right)} R_B. \tag{2.21}$$

Eq. (2.21) unfortunately yields no such simple relation as in (2.4) (i.e., $\zeta = R$). For conditions close to neutral, however, $\Psi_{M,H} \to 0$ and

$$\zeta \to \zeta_N \equiv R_B \ln \frac{z}{z_0} . \tag{2.22}$$

As $-R_B$ increases, $\Psi_{M,H}$ increase, and $\zeta \approx \zeta_N$ becomes increasingly inaccurate. Nevertheless, the near-neutral result $\zeta \approx \zeta_N$ is a reasonably accurate first approximation to (2.21) over a fairly wide range of R_B and z_0 .

To partially support this claim, we will examine the behavior of (2.21) for small values of $-\zeta$. From the definitions of $\varphi_{M,H}$,

$$\varphi_M = (1 - \alpha_M \zeta)^{-1/4} ,$$

$$\varphi_H = (1 - \alpha_H \zeta)^{-\frac{1}{2}} ,$$

we have

$$\varphi_M = 1 + \frac{\alpha_M}{4} \zeta + O(\zeta^2) ; \tag{2.23}$$

$$\varphi_H = 1 + \frac{\alpha_H}{2} \zeta + O(\zeta^2) \tag{2.24}$$

and

$$F_M = \ln \frac{Z}{Z_0} + \frac{\alpha_M}{4} (\zeta - \zeta_0) + O(\zeta^2, \zeta_0^2) ; \tag{2.25}$$

$$F_H = \ln \frac{Z}{Z_0} + \frac{\alpha_H}{2} (\zeta - \zeta_0) + O(\zeta^2, \zeta_0^2) . \tag{2.26}$$

If we neglect ζ_0 in comparison to ζ , (2.21) becomes,

$$\zeta \doteq \left(\ln\frac{z}{z_0} + \frac{\alpha_M}{4}\zeta\right)^2 \left(\ln\frac{z}{z_0} + \frac{\alpha_H}{2}\zeta\right)^{-1} R_B$$

$$= R_B \ln\frac{z}{z_0} + \frac{1}{2} R_B (\alpha_M - \alpha_H)\zeta + O(\zeta^2) \qquad (2.27)$$

$$\doteq R_B \ln\frac{z}{z_0} + \frac{1}{2}(\alpha_M - \alpha_H) R_B^2 \ln\frac{z}{z_0},$$

as a first - order approximation. For the Businger profiles, $\alpha_M = 15$ and $\alpha_H = 9$, and $\alpha_M - \alpha_H = 6$; for the Dyer-Hicks profiles, $\alpha_M = \alpha_H = 16$ and $\alpha_M - \alpha_H = 0$, thus, there is a fortuitous cancellation of the R_B^2 term. Table 1 compares ζ_N with the exact value of ζ for the two widely differing values $z_0 = 0.001\,m$ and $z_0 = 5.0\,m$. We see that ζ_N is a close approximation to ζ for $0 \leq -\zeta \leq 0.5$ (error ~ 1.4% for z_0=0.001 m; ~ 2.2% for $z_0 = 5.0$m). We note that $-\zeta_N$ systematically overestimates $-\zeta$ by a small amount that slowly increases with increasing $-R_B$ and decreasing z/z_0.

The contributions of z_0 to Ψ are generally small; we can define $\psi(z/L)$ as

$$\psi_{M,H} \equiv \lim_{z_0 \to 0} \Psi_{M,H}(z/L\,;\, z_0/L) \qquad (2.28)$$

Table 1. Exact and approximate (Eq. 2.22) values of ζ for $1 \leq -L \leq 10000$ (meters) and $0.001 \leq z_o \leq 5$ (meters). The exact values of -L and $-\zeta$ are given in columns one and two. Columns three to five are the approximate values of ζ for z_o = 0.001 (a), 0.1 (b), 5m (c).

$-L$	$-\zeta$	$-\zeta_N^{(a)}$	$-\zeta_N^{(b)}$	$-\zeta_N^{(c)}$
10000	0.005	.500 - 2	.500 - 2	.500 - 2
5000	0.010	.100 - 1	.100 - 1	.100 - 2
1000	0.05	.500 - 1	.501 - 1	.501 - 1
500	0.10	.100	.100	.100
200	0.250	.250	.253	.252
100	0.50	.507	.511	.507
50	1.0	1.02	1.04	1.02
25	2.0	2.08	2.11	2.04
10	5.0	5.32	5.45	5.13
5	10.0	10.9	11.2	10.3
1	50.0	58.4	58.3	51.4

so that

$$\Psi_{M,H} \approx \psi_{M,H} \tag{2.29}$$

for most practical purposes. This approximation is standard in the boundary layer literature; it justifies the approximation,

$$\zeta \doteq R_B \frac{\left[\ln \frac{z}{z_0} - \psi_M(\zeta)\right]^2}{\left[\ln \frac{z}{z_0} - \psi_H(\zeta)\right]} \, . \tag{2.30}$$

The exact form is of course,

$$\zeta = R_B \frac{\left[\ln \frac{z}{z_0} - \psi_M(\zeta) + \psi_M(\zeta_0)\right]^2}{\left[\ln \frac{z}{z_0} - \psi_H(\zeta) + \psi_H(\zeta_0)\right]} \, . \tag{2.31}$$

The latter expression is used in the current MRF.

B. <u>Formulas for small -Z/L</u>

Having affirmed the approximate equality of ζ and ζ_N for $-\zeta_N < 0.5$, we shall verify that $\psi_{M,H}$ can be fairly well approximated for $0 \leq -\zeta \leq 0.5$ by

$$\psi \doteq \frac{(a_0 + a_1 \zeta)}{1 + b_1 \zeta} \zeta . \tag{2.32}$$

To determine the values of a_0, a_1, b_1 we expand (2.32) in powers of :

$$\psi \doteq (a_0 \zeta + a_1 \zeta^2)(1 + b \zeta)^{-1}$$

$$\approx (a_0 \zeta + a_1 \zeta^2)(1 - b \zeta + b^2 \zeta^2)$$

$$\approx a_0 \zeta + (a_1 - a_0 b_1) \zeta^2 + (a_0 b_1^2 - a_1 b_1) \zeta^3 . \tag{2.33}$$

The expansion of Ψ_M to second order yields

$$\Psi_M = \ln\frac{z}{z_0} - F_M = \ln\frac{z}{z_0} - \int_{\zeta_0}^{\zeta} \frac{d\zeta'}{\zeta'}(1-16\zeta')^{-1/4}$$

$$= \ln\frac{z}{z_0} - \int_{\zeta_0}^{\zeta} \frac{d\zeta'}{\zeta'}\left[1 + 4\zeta' + 40\zeta'^2 + O(\zeta'^3)\right]$$

$$\doteq -4\zeta - 20\zeta^2, \tag{2.34}$$

in which ζ_0 and $O(\zeta^3)$ have been neglected. A term-by-term comparison of (2.33) with (2.34) shows that $a_0 = -4$ and $a_0 b_1 - a_1 = 20$. The coefficients a_1, b_1 can be determined by expanding Ψ_M to the ζ^3 term, thereby producing a Padé approximation (Bender and Orszag, 1978). We choose, however, to determine a_1, b_1 by forcing collocation with Ψ_M at $\zeta = -0.5$: $\Psi_M(-0.5) = 0.793358$. The result is

$$\psi_M(\xi) \doteq \frac{-4\xi + 5.140\xi^2}{1 - 6.285\xi} \qquad 0 \leq -\xi \leq 0.5 \quad . \tag{2.35}$$

A similar calculation for $\psi_H(-0.5) = 1.38629$ gives,

$$\psi_H(\xi) \doteq \frac{-8\xi + 9.563\xi^2}{1 - 7.195\xi} \qquad 0 \leq -\xi \leq 0.5 \quad . \tag{2.36}$$

Eqs. (2.35, 36) approximate $\psi_{M,H}(\xi)$ for $0 \leq -\xi \leq 0.5$ within about 10%. Further accuracy can be had, although it is probably not needed, by forcing higher-order collocation. By requiring collocation at $-\xi = 0, 0.05, 0.10$, 0.25 and .50, we can derive the approximations

$$\psi_M(\xi) = \frac{(-3.9747 + 12.3218\xi)}{1 - 7.7549\xi + 6.0413\xi^2}\,\xi \;, \tag{2.37}$$

and

$$\psi_H(\zeta) = \frac{(-7.9409 + 24.7496\,\zeta)}{(1 - 8.7051\,\zeta + 7.8993\,\zeta^2)}\,\zeta . \qquad (2.38)$$

In computing $F_{M,H}$ for $\zeta = -0.5$ and, say, $z = 50\,m$, $z_0 = 0.1\,m$, we see that $\ln \frac{z}{z_0} = 6.2$ and $\psi_M = 0.79$, $\psi_H = 1.4$. That is, ψ_M and ψ_H represent fairly modest perturbations of the more dominant $\ln z/z_0$ term. As the instability and roughness increase, the logarithmic term loses its dominance.

C. Formulas for large -Z/L

We shall now take up the problems encountered when we deal with strong instability, that is, $-\zeta \geq 0.5$. We first derive simple asymptotic expansions for $\psi_{M,H}$ that are valid for the general case $\varphi = (1 - \gamma\zeta)^{-\beta}$.

The method is useful because it clearly shows the behavior of $\psi_{M,H}$ and because it can be applied to cases in which $\beta_{M,H}$ are not simple fractions (example: Dyer's earlier relations that use $\beta_M = 0.275$, $\beta_H = 0.55$). We discuss the problems that can arise for $-\zeta >> 1$ for both the approximate and exact solutions in Section D.

To develop simple approximate formulas for $\psi_{M,H}$ that are valid for

$-\xi \gtrsim 0.5$, we decompose $\psi(\xi)$ into two integrals:

$$\psi(\xi) = \lim_{z_0 \to 0^+} \int_{z_0}^{z} \frac{dz'}{z'} \left[1 - (1 - \gamma z'/L)^{-\beta} \right]$$

$$= \lim_{z_0 \to 0^+} \left[\int_{z_0}^{\infty} - \int_{z}^{\infty} \right] \left[1 - (1 - \gamma z'/L)^{-\beta} \right] \frac{dz'}{z'}$$

$$= \lim_{x_0 \to 0^+} \left[\int_{x_0}^{\infty} - \int_{x}^{\infty} \right] \left[1 - (1 + x')^{-\beta} \right] \frac{dx'}{x'} \quad (2.39)$$

where $x \equiv -\gamma z/L$.

$\psi(\xi)$ can also be written as

$$\psi(x) = I_1(x_0) - I_2(x) \quad (2.40)$$

in which

$$I_1 = \lim_{r \to \infty} \lim_{x_0 \to 0} \hat{I}_1 \; ; \; I_2 = \lim_{r \to \infty} \hat{I}_2 \tag{2.41}$$

where

$$\hat{I}_1(x_0; r) \equiv \int_{x_0}^{r} \frac{dx'}{x'} \left[1 - (1 + x')^{-\beta} \right] \tag{2.42}$$

and

$$\hat{I}_2(x; r) \equiv \int_{x}^{r} \frac{dx'}{x'} \left[1 - (1 + x')^{-\beta} \right] \tag{2.43}$$

We first focus our attention upon $\hat{I}_1$. We add and subtract $x^{-1} e^{-x}$ to $\hat{I}_1$, in order to create integrals J_a , J_b :

$$I_1 = \lim_{r\to\infty} \lim_{x_0\to 0} \int_{x_0}^{r} \frac{1-(1+x')^{-\beta}}{x'}\, dx'$$

$$= J_a + J_b = \lim_{r\to\infty} \lim_{x_0\to 0} \int_{x_0}^{r} \left(\frac{1}{x'} - \frac{e^{-x'}}{x'} \right) dx'$$

$$+ \lim_{r\to\infty} \lim_{x_0\to 0} \int_{x_0}^{r} \left[\frac{e^{-x'} - (1+x')^{-\beta}}{x'} \right] dx' . \qquad (2.44)$$

The presence of $x^{-1} e^{-x}$ prevents $\hat{I}_1$ from diverging logarithmically as $x_0 \to 0$, but it will not thwart the divergence of J_a as $r \to 0$. We recognize the second term in J_a as the defining expression for the exponential integral function, E_1 (u) (Arfken, 1985),

$$E_1(u) \equiv \int_{u}^{\infty} \frac{e^{-t}}{t}\, dt , \qquad (2.45)$$

that for small u is given by the series

$$E_1(u) = -\tilde{\gamma} - \ln u + \sum_{m=1}^{\infty} \frac{(-1)^m u^m}{m\, m!} \tag{2.46}$$

Here $\tilde{\gamma}$ is the Euler-Masheroni constant defined by

$$\tilde{\gamma} = \lim_{m\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{m}\right) - \ln m \tag{2.47}$$

$\doteq 0.577\ 215\ 665$. Note that both the harmonic series $1 + \frac{1}{2} + \frac{1}{3} +$ . . . and $\ln m$ diverge as $m \to \infty$, but that their difference is finite. Note also that in J_a the limit $x_0 \to 0$ causes the exponential integral to diverge, but that the divergence cancels:

$$\begin{aligned} J_a &= \lim_{x_0\to 0} \lim_{r\to\infty} \ln(r/x_0) + \tilde{\gamma} + \ln x_0 \\ &= \lim_{r\to\infty} \ln r + \tilde{\gamma} \end{aligned} \tag{2.48}$$

We are left with one logarithmic divergence in the evaluation of J_a. This constitutes no special problem, as we shall see.

We now examine

$$J_b = \lim_{x_0 \to \infty} \int_{x_0}^{\infty} \left[\frac{e^{-x'} - (1+x')^{-\delta}}{x'} \right] dx' \tag{2.49}$$

We begin by writing the integral expression for the gamma function $\Gamma(u)$ as

$$\ln \Gamma(u) = \int_0^{\infty} dt \left[\frac{u-1}{t} e^{-t} - \frac{e^{-t} - e^{-ut}}{t(1-e^{-t})} \right] \tag{2.50}$$

which is then differentiated to get,

$$\frac{d}{du} \ln \Gamma(u) = \int_0^{\infty} \left[\frac{e^{-t}}{t} - \frac{e^{-ut}}{(1-e^{-t})} \right] dt \tag{2.51}$$

which of course is the same as $[1/\Gamma(u)][d\Gamma(u)/du]$.
This logarithmic derivative is the definition of the Euler 'diagram' or 'psi' function (Gradshteyn and Ryzhik, 1965):

$$\Psi_E(u) \equiv \frac{1}{\Gamma(u)}\frac{d\Gamma(u)}{du} = \int_0^{\infty}\left[\frac{e^{-t}}{t} - \frac{e^{-ut}}{(1-e^{-t})}\right]dt \qquad (2.52)$$

Let us now rewrite the second term of J_b

and change variables: $x \to e^{t}-1$. This change yields

$$\int_0^{\infty}\frac{1}{x}(1+x)^{-u}\,dx = \int_0^{\infty}\frac{e^{-ut}}{(1-e^{-t})}\,dt \qquad (2.53)$$

an equality that permits us to write

$$J_b = \Psi_E(\beta) = \int_0^{\infty}\left[\frac{e^{-x}-(1+x)^{-\beta}}{x}\right]dx\,. \qquad (2.54)$$

Thus, J_6 is simply equal to the Euler digamma function. Certain values of the digamma function have been tabulated; values of particular interest to us are,

$$\Psi_E(1/2) = -\tilde{\gamma} - 2\ln 2 \doteq -1.963\,510\,026$$

$$\Psi_E(1/4) = -\tilde{\gamma} - \frac{\pi}{2} - 3\ln 2 \doteq -4.227\,453\,534 \tag{2.55}$$

$$\Psi_E(1/3) = -\tilde{\gamma} - \frac{\pi}{2}\sqrt{\frac{1}{3}} - \frac{3}{2}\ln 3 \doteq -3.132\,033\,782.$$

Let us now turn our attention to $\hat{I}_2$ and I_2. To approximate these functions, we need only to expand $\hat{I}_2$,

$$\hat{I}_2 = \int_x^r \frac{1-(1+x')^{-\beta}}{x'}\,dx'$$

$$= \int_x^r \frac{dx'}{x'}\left[1 - x'^{-\beta}\left(1+\frac{1}{x'}\right)^{-\beta}\right]. \tag{2.56}$$

If we require that $x>1$, we can write

$$\hat{I}_2 = \int_x^r \frac{dx'}{x'}\left\{1 - x'^{\beta}\left[1 - \frac{\beta}{x'} - \frac{\beta(\beta+1)}{2}\frac{1}{x'^2} + \dots\right]\right\}$$

$$= \ln\frac{r}{x} - S(x;\beta) + S(r:\beta) \tag{2.57}$$

in which $S(t;\beta)$ is the series

$$S(t;\beta) = \frac{1}{\beta} t^{-\beta} - \frac{\beta}{\beta+1} t^{-(\beta+1)} - \frac{\beta(\beta-1)}{2(\beta+2)} t^{-(\beta+2)} \tag{2.58}$$

$$+ \ldots .$$

For $\varphi_H(z/L)$ with $\beta = 1/2$ we have,

$$S(x;1/2) = 2x^{-1/2} - \frac{1}{3} x^{-3/2} + \frac{1}{20} x^{-5/2} + \ldots . \tag{2.59}$$

For $\varphi_M(z/L)$ with $\beta = 1/4$ we get,

$$S(x;1/4) = 4x^{-1/4} - \frac{1}{5} x^{-5/4} + \frac{1}{24} x^{-9/4} + \ldots . \tag{2.60}$$

The presence of $\ln r$ in $\hat{I}_2$ is precisely what we need to cancel the divergence in J_a. Our final result is the simple expression

$$\Psi(x) = \ln x + S(x;\rho) + \tilde{\gamma} + \Psi_E(\rho) . \tag{2.61}$$

Although we have been less than circumspect in our manipulation of limits and divergences, our final expressions can be justified -- they yield correct results.

Explicit expressions for $\Psi_M(\xi)$ and $\Psi_H(\xi)$ with $x = -16Z/L$ are given by

$$\Psi_M \sim -3.650\,237\,869 + \ln x$$

$$+ 2\frac{1}{(-\xi)^{1/4}} - \frac{1}{160}\frac{1}{(-\xi)^{5/4}} + \frac{1}{12{,}288}\frac{1}{(-\xi)^{9/4}} \tag{2.62}$$

$$+ \cdots ,$$

and

$$\Psi_H \sim -1.386\,294\,361 + \ln x$$

$$+\frac{1}{2}\frac{1}{(\xi)^{1/2}}-\frac{1}{144}\frac{1}{(-\xi)^{3/2}}+\frac{1}{20{,}480}\frac{1}{(-\xi)^{5/2}} \qquad (2.63)$$

$$+\ldots,$$

or

$$\psi_M(\xi) \sim -0.877\,649\,147 + \ln(-\xi)$$

$$+2\frac{1}{(-\xi)^{1/4}}-\frac{1}{160}\frac{1}{(-\xi)^{5/4}}+\frac{1}{12{,}288}\frac{1}{(-\xi)^{9/4}}+\ldots, \qquad (2.64)$$

$$\psi_H(\xi) \sim 1.386\,294\,361 + \ln(-\xi)$$

$$+\frac{1}{2}\frac{1}{(-\xi)^{1/2}}-\frac{1}{144}\frac{1}{(-\xi)^{3/2}}+\frac{1}{20{,}480}\frac{1}{(-\xi)^{5/2}}+\ldots. \qquad (2.65)$$

Relations (2.64) and (2.65), along with shorter versions in which first the $(-\xi)^{-5/2}$ term and then the $(-\xi)^{-3/2}$ terms are omitted, are shown in Tables 2 and 3. As expected, the accuracy of the asymptotic relations improves as $(-\xi)$ increases and if the $(-\xi)^{-3/2}$ and $(-\xi)^{-5/2}$ terms are included in the sum. We see that $\psi^{(3)}_{M,H}$ are accurate within about 1.7% or less for $-\xi \geq 0.25$, and $\psi^{(1)}_{M,H}$ within about 1.8% or less for $-\xi > 0.50$. Accordingly, $\psi^{(1)}_{M,H}$ will be adopted as adequate approximations to $\psi_{M,H}$ for strong instability, that is, for $-\xi > 0.50$.

Table 2. Comparison of the exact values of $\Psi_M(\xi)$ with the approximate values calculated from Eq. (2.64) with the $(-\xi)^{-9/4}$ $(= \Psi_M^{(3)})$; $(-\xi)^{-5/4}$ $(= \Psi_M^{(2)})$; $(-\xi)^{-1/4}$ $(= \Psi_M^{(1)})$ terms.

$-\xi$	Ψ_M^{exact}	$\Psi_M^{(3)}$	$\Psi_M^{(2)}$	$\Psi_M^{(1)}$
0.005	0.01952	8.886	-3.356	1.345
0.01	0.03815	1.439	-1.135	0.8417
0.05	0.1636	0.1606	0.09176	0.3561
0.1	0.2836	0.2797	0.2652	0.3763
0.25	0.5319	0.5310	0.5291	0.5645
0.5	0.7934	0.7931	0.7928	0.8076
1.0	1.116	1.116	1.116	1.112
2.0	1.495	1.495	1.495	1.497
5.0	2.068	2.068	2.068	2.069
50.	3.786	3.786	3.786	3.786

Table 3. Comparison of the exact values of $\Psi_M(\xi)$ with the approximate values calculated from Eq. (2.65) with the $(-\xi)^{-9/4} \left(= \psi_H^{(3)}\right)$; $(-\xi)^{-5/4} \left(= \psi_H^{(2)}\right)$; $(-\xi)^{-1/4} = \left(= \psi_H^{(1)}\right)$ terms.

$-\xi$	ψ_H^{exact}	$\psi_H^{(3)}$	$\psi_H^{(2)}$	$\psi_H^{(1)}$
0.005	0.03885	11.14	-16.48	3.159
0.01	0.07559	-0.2805	- 5.163	1.781
0.05	0.3154	0.09285	0.005500	0.6266
0.1	0.5343	0.4607	0.4452	0.6648
0.25	0.9624	0.9460	0.9444	1.000
0.50	1.386	1.381	1.381	1.400
1.0	1.881	1.879	1.879	1.886
2.0	2.431	2.431	2.431	2.433
5.0	3.219	3.219	3.219	3.219
50.	5.369	5.369	5.369	5.369

D. Preparing the Formulas for Use in NWP Models

The boundary layer relations that we have examined must be applicable to regions of extreme variations in instability and roughness in order to be useful in NWP models. NWP models often exhibit far stronger variations in z_o and L than would be expected in boundary layer field experiments. While it is expecting too much for standard boundary layer relations to yield accurate results for all modeling conditions, it is reasonable to expect that the relations will at least not degrade model performance.

Fortunately, the application of the new relations is usually straight-forward. The steps are:

1) R_B is calculated from (2.5).

2) ζ_N is calculated from (2.22).

3) $\psi_M(\zeta_N)$ and $\psi_H(\zeta_N)$ are computed from (2.37) and (2.38) for $-\zeta_N \leq 0.5$ and from (2.64) and (2.65) for $-\zeta_N \geq 0.5$.

4) Approximate F_M and F_H from

$$F_{M,H} \doteq \ln \frac{h}{z_0} - \psi_{M,H}(\xi_N). \tag{2.66}$$

5a) Estimate u_* and θ_* from (2.8) and (2.9).

5b) Alternatively, compute the momentum, heat, and humidity transfer coefficients from

$$C_M = k^2 F_M^{-2} \; ; \; C_{H,Q} = k^2 (F_M F_H)^{-1}. \tag{2.67}$$

6a) Compute the fluxes $\mathcal{F}_M$ and $\mathcal{F}_H$ using (1.7) and (1.8). The latent heat flux $\mathcal{F}_Q$ is calculated from (1.9) and the relation,

$$q_* = \frac{\theta_*}{\Delta\theta} \Delta q. \tag{2.68}$$

This formula for q_* follows from $\mathcal{C}_Q = \mathcal{C}_H$ and the extension of (2.8) and (2.9) to Δq.

6b) Alternatively, $\mathcal{F}_{M,H,Q}$ can be calculated from

$$\mathcal{F}_M = -\rho\, C_M\, U^2$$

$$\mathcal{F}_H = -\rho\, C_p C_H\, U\, \Delta\theta \tag{2.69}$$

$$\mathcal{F}_Q = -\rho\, \mathcal{L}\, C_H\, U\, \Delta q .$$

Steps (5a, 6a) and (5b, 6b) give the same results and merely differ in form.

By contrast, the MRF follows this procedure:

I.) The same as 1.)

II.) ξ is calculated by iteration. The 'exact' relation (2.21) is solved using Newton's method to an 'accuracy' that is substantially greater than needed.

III.) $\Psi_{M,H}(\xi, \xi_0)$ are computed from $\Psi_{M,H}(\xi,\xi_0) = \psi_{M,H}(\xi) - \psi_{M,H}(\xi_0)$ which uses (2.19) and (2.20). Note that $\psi_{M,H}(\xi_0)$ are not neglected.

IV.) $C_M(\xi, \xi_0)$ and $C_H = C_Q(\xi, \xi_0)$ are computed from

$$C_M(\zeta, \zeta_o) = k^2\left[\ln\left(\frac{h}{z_o}\right) - \Psi_M(\zeta, \zeta_o)\right]^{-2} \quad (2.70)$$

$$C_H(\zeta, \zeta_o) = k^2\left[\ln\left(\frac{h}{z_o}\right) - \Psi_M(\zeta, \zeta_o)\right]^{-1}\left[\ln\left(\frac{h}{z_o}\right) - \Psi_H(\zeta, \zeta_o)\right]^{-1} \quad (2.71)$$

We see that $\psi_{M,H}$ and $\Psi_{M,H}$ increase as $-\zeta$ increases; accordingly, $C_{M,H,Q}$ increase without limit with increasing instability.

E. Physical and Computational Limitations

The steps outlined above prompt questions about the limits of accuracy of the approximations as well as the limits of suitability of the 'exact' relations. The first question concerns the capacity of either the approximate or 'exact' solutions to give reasonable fluxes where h is not much larger than z_o. The second question concerns the fidelity of the exact solution and the possible breakdown of the approximate solution when $-\zeta$ is large $(\gg 1)$. The third question is: Of what use is the approximate solution when $-L$ is not much larger than z_o (and perhaps even smaller than z_0). These questions are related, of course, and they have no completely satisfactory answers at present. We shall suggest several rather pragmatic provisional solutions.

First question. It is known that the logarithmic wind law is not valid for $z \sim z_0$. It is also known that for both smooth and moderately rough surfaces with $z/z_0 \gtrsim 100$ that the logarithmic wind law is valid for both laboratory (wind tunnel) and atmospheric measurements. Following Garratt (1980), we denote by z_* the lowest z for which the profile laws are approximately valid for neutral and unstable lapse rates. Tennekes (1973) has suggested for the neutral case that $z_* \approx 10-100\, z_0$. The region $z < z_*$ is Garratt's 'roughness sublayer'.

At heights much lower than z_* the individual roughness elements can be 'sensed' as the result of turbulent wakes created by flow around individual elements. This violates the similarity conditions that lead to the conventional flux-profile law by forcing a length scale (z_*) in addition to L be used in determining $\varphi_{M,H}$. Wind tunnel data suggest that the wakes of individual elements propagate to heights several times (say, ~4) the heights (h_0) of the average roughness element. If we use as a crude rule-of-thumb that $z_0 \sim h_0/10$, then the minimum height for the profile laws to be valid is $z_* \sim 40\, z_0$.

Garratt's (1980) analysis of data from two "flat, very rough, tree-covered terrain" sites suggests that $z_* \approx 35\, z_0$ for momentum and that $z_* \approx 100\, z_0$ for heat. This is a disconcerting result. It means that if the largest roughness lengths in the MRF are, say, $z_0 \approx 10\,m$, then the standard profile laws do not apply to heights below ≈350 m for momentum and≈1000 m for heat (and, presumably, humidity). Both heights are as great as or greater than the heights of the lowest (surface) layers of most NWP models and render invalid, or at least call into question, the surface fluxes.

In our calculations, we shall take a less restrictive position and assume that the surface profile laws are valid provided:

$h > 10\, z_0$. If we encounter points where $z_0 > h/10$, then we will decrease z_0 so that we satisfy the condition $z_0 = h/10$.

Second question. The Businger-Dyer-Hicks profile laws are thought to be reasonably accurate for $-z/L \lesssim 3$. Model computations, however, can create unstable conditions that significantly exceed this moderate limitation. For these cases, computed fluxes can be completely unrealistic.

Consider the following situation: z_0 , Δq , $\Delta\theta$, and h are held constant while U is progressively decreased. Decreasing U to U $\rightarrow$ o rapidly increases $-R_B$ and $-\xi$. When $-\xi$ increases, Ψ_M and Ψ_H increase (see Tables 2 and 3). This decreases F_M and F_H which, in turn, increases C_H and C_M (see 2.70, 2.71). For $|L| >> z_0$, the decrease in U exceeds the increase in $C_{M,H,Q}$, thereby decreasing the fluxes of momentum, heat, and humidity. This situation is typified by Case A of Table 4. The near neutral condition, $-L = 1000$ m , with the strongest wind ($\doteq 24$ m/s), produces the strongest heat flux in Case A. The most unstable condition of Case A, $-L = 1$ m , with the weakest wind ($\doteq 0.69$ m/s), produces the weakest flux. (Note: the heat flux in K m/s can be converted to W/m^2 by multiplying by $\doteq 1.2 \times 10^3$).

Case A with $z_0 = 0.01$ m is typical of boundary layer field sites. Cases B, C, and D with $z_0 = 1$ m , 10 m

Table 4. Momentum (u_*^2) and temperature fluxes ($-u_*\theta_*$) for a wide range of wind speeds, temperature differences, and roughness lengths.

A

U (m/s) =	23.9	7.48	5.25	3.68	2.29	1.60	0.687
$-L$ (m) =	1000	100	50	25	10	5	1
$-z/L$ =	0.05	0.5	1.0	2	5	10	50
$-z_0/L$ =	0.00001	0.0001	0.0002	0.0004	0.001	0.002	0.002
u_*^2 (m^2/s^2) =	1.30	0.150	0.086	0.0439	0.0202	0.0114	0.0332
$-u_*\theta_*$ (K m/s) =	0.111	0.0434	0.0342	0.0275	0.0214	0.0182	0.0143

B

	16.2	5.06	5.56	2.50	1.57	1.11	0.493
	1000	100	50	25	10	5	1
above	0.05	0.5	1.0	2	5	10	50
	0.001	0.01	0.02	0.04	0.100	0.20	1.0
	2.97	0.411	0.246	0.153	0.0872	0.0589	0.0252
	0.382	0.197	0.182	0.180	0.192	0.214	0.299

C

	10.4	3.28	2.31	1.63	1.03	0.729	0.326
	1000	100	50	25	10	5	1
above	0.05	0.5	1.0	2	5	10	50
	0.01	0.10	0.20	0.40	1.0	2.0	10.0
	7.81	1.41	0.935	0.637	0.394	0.276	0.123
	1.63	1.26	1.35	1.52	1.85	2.17	3.21

D

	12.4	3.85	2.70	1.90	1.18	0.838	0.373
	1000	100	50	25	10	5	1
above	0.1	1.0	2	4	10	20	100
	0.001	0.01	0.02	0.04	0.1	0.20	1.0
	1.31	0.191	0.115	0.0724	0.0414	0.0280	0.0120
	0.112	0.0625	0.0586	0.0583	0.0629	0.0701	0.0986

A: $z_0 = 0.01$ m ; $\Delta\theta = -2$K ; $h = 50$ m

B: $z_0 = 1$ m ; $\Delta\theta = -2$K ; $h = 50$ m

C: $z_0 = 10$ m ; $\Delta\theta = -2$K ; $h = 50$ m

D: $z_0 = 1$ m ; $\Delta\theta = -1$K ; $h = 100$ m are examples of areas of extreme roughness found in NWP models. It is with Cases B, C, and D that we encounter computational and physical difficulties.

In all cases the surface stress decreases with increasing $-h/L$. In sections B, C, D we observe a c urious behavior, however, as the windspeed decreases, the heat flux decreases, reaches a minimum, and then increases. This is counter-intuitive. Case C with $z_o = 10 m$ is particularly extreme. The lowest wind speed, $U \doteq 0.33 m/s$, produces the highest heat flux, $3.2 K m/s$ ($\approx 3900 W m^{-2}$), about three times the solar constant. We seek to eliminate these runaway heat fluxes. Although it may appear that Ψ_M and Ψ_H, (both increasing functions), eventually equal and then exceed $\ln h/z_o$, causing the fluxes to become infinite and then reversing sign, this is not the case. From (2.6 - 2.9), it follows that $\Psi_{M,H}$ approach, but never equal or exceed, zero.

On the otherhand, $\psi_{M,H}$ are only approximations to $\Psi_{M,H}$. For large enough $-\varsigma$, $\psi_{M,H}$ will equal and then exceed $\ln h/z_o$, causing the computed fluxes to blow-up and then reverse sign. These nonsensical and useless results can be prevented, as we will see.

Consider the smallest allowed value of h/z_o, that is,

$h/z_0 = 10$. Now, Ψ_H increases faster than Ψ_M . For very small $-\xi$: $\Psi_H = -8\xi$, $\Psi_M = -4\xi$ (see 2.35, 2.36). For very large $-\xi$: $\Psi_H \sim$ Eq. (2.65), $\Psi_M \sim$ Eq. (2.64). Since for large $-\xi$, and noting that $\ln(h/z_0)_{min} = \ln 10 = 4.3$ and also that $\Psi_H \sim 1.39 + \ln(-\xi) + \frac{1}{2}(-\xi)^{-1/2}$, setting Ψ_H equal to $[\ln(h/z_0)]_{min} = \Psi_H(\xi)]_{max}$, we get $\xi = -1.7$ This means $-\xi = 1.7$ is the largest value of instability we can use and be assured that C_H (Eq. 2.67) does not blow-up. This limitation is unacceptable.

There are several artifices that can be employed to avoid some of these problems. None of these devices can be fully justified, however, by appealing to field data. One artifice invokes free convection. There are at least four ways that this can be done. First, Deardorff's (1972) convective velocity scale w_* can be 'switched on' whenever the values of $C_M \gtrsim 4\, C_M$ (neutral) and $C_H \gtrsim 6.6\, C_H$ (neutral). The convective velocity scale is $w_* = (g\mathcal{F}_H H/\rho c_p \bar{\theta})^{1/3}$ in which H is the height of the convective boundary layer. Second, free convection can be introduced by fiat by forcing the velocity dependence to drop out of the calculation of $\mathcal{F}_{H,Q}$ whenever $-\xi$ or $-R_B$ exceeds a specified number. Third, the Businger-Dyer-Hicks flux-profile laws can be used for $-\xi \leqslant c_c =$ constant, where c_c is on the order of unity. For $-\xi > c_c$, we use

$\varphi_M = -b_M \xi^{-1/3}$ and $\varphi_H = -b_H \xi^{-1/3}$, in which $b_{M,H}$ are positive constants. Fourth, flux-profile laws can be used that embrace free convection as an asymptotic limit. The KEYPS (or O'KEYPS) equation is a well known example of this.

Invoking Deardorff's mixed layer scaling to limit runaway $\mathcal{F}_{H,Q}$ involves several assumptions relating to $\mathcal{F}_{H,Q}$. It is assumed that U can be approximated by $\approx .7 w_*$, and that $C_M^* \equiv 4 \; C_M$ (neutral) and $C_H^* = 6.6 C_H$ (neutral). This leads to

$$\mathcal{F}_H = \rho C_p C_H^* (gH/\bar{\theta})^{1/2} |\Delta \theta|^{3/2}$$

$$\mathcal{F}_M = \frac{1}{2} \rho C_M^* (g H \mathcal{F}_H / \rho C_p \bar{\theta})^{2/3} \qquad (2.72)$$

We see that the surface heat flux depends upon $|\Delta\theta|^{3/2}$ and that there is no longer a need to compute $R_B, \xi, \Psi_{M,H}$ and $F_{M,H}$.

Introducing free convection by fiat is rather direct. We assume that

$$\mathcal{F}_H \sim A \rho C_p U \Delta\theta (-R_B)^M \qquad (2.73)$$

where A and M are constants. To force U to vanish, we choose M=1/2. The result is

$$F_H \sim A \rho C_p \left(\frac{g h}{\bar{\theta}}\right)^{1/2} |\Delta \theta|^{3/2} \qquad (2.74)$$

Deardorff's result is similar, except H and h are interchanged. Carson (1982) also gives a free-convection result. For $h \gg z_0$, his expression is

$$F_H \sim A' \rho C_p \left(\frac{g z_0}{\bar{\theta}}\right)^{1/2} |\Delta \theta|^{3/2} . \qquad (2.75)$$

This result shares the $|\Delta \theta|^{3/2}$ dependence with the two previous expressions. The difference is the length scale z_0. The third method takes note of Wyngaard, <u>et al.</u> 's remark that $\varphi_H(\xi) \approx 0.23(-\xi)^{-1/3}$ (see 1.18) for $0.5 \le -\xi \le 2$. We speculate that this $-1/3$ law also holds for all $-\xi \geq 2$ and also that $\varphi_M(\xi) \sim b(-\xi)^{-1/3}$ holds for $-\xi \geqslant 0.5$. After integration from z_0 to h, we have

$$\Psi_{M,H}(\xi,\xi_0) = \int_{z_0}^{z=h} \frac{dz'}{z'}\left[1 - \varphi_{M,H}(z'/L)\right]$$

$$= \Psi_{M,H}\left(-0.5; -0.5\frac{z_0}{z}\right) + \ln(-\xi) + \ln 2 \qquad (2.76)$$

$$+ 3\, b_{M,H}\left[\frac{1}{(-\xi)^{1/3}} - \frac{1}{(.5)^{1/3}}\right], \quad -\xi > 0.5 ;$$

where $b_{M,H}$ = consts .

Our method for dealing with large $-\xi$ and z_0 will not involve invoking free convection but by simply limiting $\Psi_{M,H}(\xi)$ and $\Psi_{M,H}(\xi_0)$.

We first note Table 4 does not seem to indicate a minimum value of $-u_*\theta_*$ for any particular value of U, R_B, $\Delta\theta$ h or ξ . However, by examining many examples, a pattern begins to emerge. A minimum occurs for small -L and large z_0 wherever

$$m z_0 \geqslant -L$$

when $m \approx 20 - 50.$ (2.77)

That is, a minimum can only appear when z_0 begins to approach the size of -L,

$$-z_0/L \geqslant \frac{1}{m}. \qquad (2.78)$$

To avoid the problem of runaway heat and humidity fluxes, we choose

$$\max|z_0/L| = \frac{1}{50}. \qquad (2.79)$$

This choice has the added advantage of assuring us that $\psi_{M,H}(\zeta_0)$ can be neglected in comparison to $\psi_{M,H}(\zeta)$.

To achieve the restriction on $|z_0/L|$, we simply increase -L. We do this by increasing U so that $|z_0/L| = \frac{1}{50}$ is satisfied. This change decreases $C_{H,Q}$ so that the possibility of runaway heat and moisture fluxes is avoided.

F. Summary

A simple, economical method has been derived for computing the surface layer fluxes of momentum, heat, and humidity. The functions involved in the computation have been simplified and split into two categories. The categories correspond to: weak and mild instability; strong instability. Padé - like functions have been derived for the first case. Asymptotic functions have been derived for the strongly unstable case.

To avoid unreasonable fluxes, two restrictions are placed on the use of the method. The first restriction is invoked when $z \rightarrow z_*$, that is, when the surface similarity relations are forced to apply to the 'roughness sublayer'. The second restriction is applied whenever the surface layer becomes so unstable that $-L \rightarrow z_0$. Physically and computationally plausible reasons for the restrictions are presented.

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