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**Abstract.** Shear dispersion in time-varying flows is shown to be parameterizable as effective diffusion with a time-varying diffusivity. This parameterization is appropriate only for times long in comparison with the cross-shear mixing time. High frequency variations of the flow do not contribute substantially to the mixing. Low frequency variations of the flow can be accounted for by simply allowing the parameters in the expression for the effective diffusivity to take on a time dependence.

## 1. Introduction

Previous discussions (Bowden, 1965; Schönfeld, 1961; Okubo, 1967) of harmonically varying shear have concluded that high frequency time variations do not contribute substantially to the mixing. Here it is shown that, furthermore, a time-dependent diffusivity can be introduced to parameterize time-dependent shear dispersion. A simple solvable model (Thacker, 1976) is used to obtain closed form expressions for shear dispersion with arbitrary time dependence.

## 2. Closed Form Expressions

Shear dispersion is mixing due to the combined effects of velocity shear and cross-shear mixing. A simple model, which views the shear as a flow consisting of two layers of fluid moving in opposite directions with velocities $\pm u$ and with mixing of contaminant between the layers at a rate $\alpha$, can be expressed by the equations

$$
\left.\begin{aligned}
&\frac{\partial C_1}{\partial t} + u\,\frac{\partial C_1}{\partial x} = -\alpha(C_1 - C_2) + S_1 \\
\text{and}\qquad &\frac{\partial C_2}{\partial t} - u\,\frac{\partial C_2}{\partial x} = -\alpha(C_2 - C_1) + S_2
\end{aligned}\right\} , \tag{1}
$$

where $C_1(x,t)$ and $C_2(x,t)$ are distributions of contaminant in the two layers, and $S_1(x,t)$ and $S_2(x,t)$ are sources of contaminant. If $u$ and $\alpha$ do not vary in time, then $\overline{C} = \frac{1}{2}(C_1 + C_2)$ is well approximated by the solution of a diffusion equation with effective diffusivity given by $K^* = u^2/2\alpha$ (Thacker, 1976). It will be shown that the same is true even if $u$ and $\alpha$ depend upon time.

For the case of time-independent flow, where $u$ and $\alpha$ are not functions of time, closed form solutions for $\overline{C} = \frac{1}{2}(C_1 + C_2)$ and $\Delta C = \frac{1}{2}(C_1 - C_2)$ can be obtained in terms of a Green's function,

$$
G_{u,\alpha}(x,t) = \begin{cases} \dfrac{1}{2u} e^{-\alpha t} I_0\left(\left[(\alpha t)^2 - \left(\dfrac{\alpha x}{u}\right)^2\right]^{\frac{1}{2}}\right); & |x| < ut \text{ and } t > 0 \\[2ex] 0; \text{ otherwise.} & \end{cases} \tag{2}
$$

The matrix notation,

$$
C = \begin{pmatrix} \overline{C} \\ \\ \Delta C \end{pmatrix}, \qquad S = \begin{pmatrix} \overline{S} \\ \\ \Delta S \end{pmatrix}, \qquad \mathcal{G}_{u,\alpha} = \begin{pmatrix} \dfrac{\partial}{\partial t} + 2\alpha & -u\dfrac{\partial}{\partial x} \\[2ex] -u\,\dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial t} \end{pmatrix} G_{u,\alpha}\,, \tag{3}
$$

allows the solution to (1) for arbitrary initial conditions $C(x,t=0)$ and sources of contaminant $S(x,t)$ to be expressed compactly as

$$
C(x,t) = \int_{-\infty}^{+\infty} dx' \int_0^t dt'\,\mathcal{G}_{u,\alpha}(x-x',t-t')\,S(x',t') + \int_{-\infty}^{+\infty} dx'\,\mathcal{G}_{u,\alpha}(x-x',t)\,C(x',t=0). \tag{4}
$$

Equation (4) can be shown to be valid even when $u$ and $\alpha$ are functions of time. In this case, the matrix Green's function is given by

$$
\mathcal{G}_{u(t),\alpha(t)}(x,t) = \underset{N\to\infty}{\text{limit}} \int_{-\infty}^{+\infty} \prod_{i=1}^{N} dx_i\, \mathcal{G}_{u_i,\alpha_i}\!\left(x_i,\frac{t}{N}\right) \delta\!\left(x - \sum_{i=1}^{N} x_i\right) \quad . \tag{5}
$$

The matrices $\mathcal{G}_{u_i,\alpha_i}$ are the matrix Green's functions that govern the solution in $i^{\text{th}}$ time interval, from $(i-1)t/N$ to $it/N$ in which the parameters have the approximately constant values $\alpha_i$ and $u_i$. It is important that the matrix Green's functions $\mathcal{G}_{u_i,\alpha_i}$ are arranged for multiplication with the index $i$ decreasing to the right because they do not commute. Thus, $\mathcal{G}_{u(t),\alpha(t)}$ is the $N$-fold convolution of the Green's function for each of $N$ time intervals composing the total interval from 0 to $t$. In the limit, $N$ goes to infinity, so that this is an infinite convolution.

This infinite convolution is difficult to evaluate. However, it is possible to obtain closed form expressions for the moments $\langle X^n \rangle$ of the mean contaminant distribution $\bar{C}(x,t)$ for arbitrary functions $\alpha(t)$ and $u(t)$. The Fourier transform of $\bar{C}(x,t)$ is the generating function for these moments. Equation (4) expresses $\bar{C}(x,t)$ in terms of convolution of initial conditions and sources with $\mathcal{G}_{u(t),\alpha(t)}$, which in turn is an infinite convolution. Since the Fourier transform of a convolution is a product, the problem is reduced to an easier one of evaluating an infinite product.

The Fourier transform of $\mathcal{G}_{u(t),\,\alpha(t)}$ is given by

$$\Gamma_{u(t),\alpha(t)}(k,t) = \lim_{N\to\infty} \prod_{i=1}^{N} e^{-\frac{\alpha_i t}{N}} \begin{bmatrix} \cos\frac{\Omega_i t}{N} + \frac{\alpha_i}{\Omega_i}\sin\frac{\Omega_i t}{N} & -ik\frac{u_i}{\Omega_i}\sin\frac{\Omega_i t}{N} \\ -ik\frac{u_i}{\Omega_i}\sin\frac{\Omega_i t}{N} & \cos\frac{\Omega_i t}{N} - \frac{\alpha_i}{\Omega_i}\sin\frac{\Omega_i t}{N} \end{bmatrix}, \quad (6)$$

where $\Omega_i = (u_i^2 k^2 - \alpha_i^2)^{\frac{1}{2}}$. The moments of the distribution can be obtained by multiplying moment matrices, which are generated according to

$$\langle X^n \rangle = \left(\frac{1}{-i}\right)^n \frac{\partial^n}{\partial k^n} \Gamma_{u(t),\alpha(t)}(k,t)\bigg|_{k=0}, \quad (7)$$

with the Fourier transform of the initial distribution, if there are no sources. The first few moment matrices are found to be

$$
\left.
\begin{aligned}
\langle X^0\rangle &= \begin{bmatrix} 1 & 0 \\ 0 & \exp\left[-2\int_0^t dt'\alpha(t')\right] \end{bmatrix} \\
\langle X^1\rangle &= \begin{bmatrix} 0 & \int_0^t dt'u(t')\exp\left[-2\int_0^{t'} dt''\alpha(t'')\right] \\ \int_0^t dt'u(t')\exp\left[-2\int_{t'}^{t} dt''\alpha(t'')\right] & 0 \end{bmatrix} \\
\langle X^2\rangle &= \begin{bmatrix} 2\int_0^t dt'\int_0^{t'} dt''u(t')u(t'')\exp\left[-2\int_{t''}^{t'} dt'''\alpha(t''')\right] & 0 \\ 0 & 2\int_0^t dt'\int_0^{t'} dt''u(t')u(t'')\exp\left[-2\int_0^{t''}+\int_{t'}^{t} dt'''\alpha(t''')\right] \end{bmatrix}
\end{aligned}
\right\} . \qquad (8)
$$

Consider the case of initially localized distributions, $\overline{C}(x,t=0)=\delta(x)$ and $\Delta C(x,t=0)=\beta\delta(x)$, $|\beta|\leq 1$. If $\beta=0$, the contaminant is initially distributed equally between the two layers; if $\beta=\pm 1$, it lies entirely within one of the two layers. Also, suppose that there are no continuous sources of contaminant $\overline{S}(x,t)=\Delta S(x,t)=0$. Then the first few moments of $\overline{C}(x,t)$,

$$
\langle x^n\rangle = \int_{-\infty}^{+\infty} dx\, x^n\, \overline{C}(x,t), \qquad (9)
$$

are given by

$$
\left.
\begin{aligned}
\langle x^0\rangle &= 1 \\
\langle x^1\rangle &= \beta \int_0^t dt'u(t')\exp\left[-2\int_0^{t'} dt''\alpha(t'')\right] \\
\langle x^2\rangle &= 2\int_0^t dt'\int_0^{t'} dt''u(t')u(t'')\exp\left[-2\int_{t''}^{t'} dt'''\alpha(t''')\right]
\end{aligned}
\right\} . \qquad (10)
$$

These moments can be evaluated for arbitrary time dependence of the shear and mixing. The zeroth moment indicates that the total amount of contaminant is constant in time, regardless of variations of the flow. The first moment indicates that there is a horizontal asymmetry only if initially there is a vertical asymmetry. However, the magnitude of the horizontal asymmetry depends upon the time dependence of the flow parameters, $u$ and $\alpha$. The second moment measures the square of the width of the distribution.

## 3. Parameterization as Time-Dependent Diffusion

A diffusion process with time-dependent diffusivity $K^*(t)$ would yield the moments

$$\left.\begin{aligned} \langle x^0 \rangle &= 1 \\ \langle x^1 \rangle &= 0 \\ \langle x^2 \rangle &= 2\int_0^t dt' K^*(t') \end{aligned}\right\} . \tag{11}$$

Comparison of the second moment in (10) with that in (11), shows that in some sense the expression

$$f(t) = u(t)\int_0^t dt' u(t') \exp\left[-2\int_{t'}^t dt'' \alpha(t'')\right] \tag{12}$$

must approach $K^*$ $(t)$ if the time-dependent shear effect is to behave like time-dependent diffusion. This becomes more clear after consideration of the following cases for different time dependencies of $u$ and $\alpha$.

**Case One.** In the case of no time dependence, where $u$ and $\alpha$ are constant,

$$\left.\begin{aligned} \langle x \rangle &= \beta \frac{u}{2\alpha}\left(1 - e^{-2\alpha t}\right) \\ \langle x^2 \rangle &= \frac{u^2}{\alpha} t\left[1 - \frac{1}{2\alpha t}\left(1 - e^{-2\alpha t}\right)\right] \\ f(t) &= \frac{u^2}{2\alpha}\left(1 - e^{-2\alpha t}\right) \end{aligned}\right\} . \tag{13}$$

The time dependence of $f$ should not be interpreted as time-dependent diffusion. Shear dispersion, for this case, can be parameterized as diffusion only for times long in comparison with the cross-shear mixing time; i.e., $\alpha t \gg 1$. In this limit $K^* = f = \frac{u^2}{2\alpha}$, $\langle x^2 \rangle = 2(u^2/2\alpha)t$, and $\langle x^2 \rangle \gg \langle x \rangle^2$ (Thacker, 1976).

**Case Two.** In the case of harmonically varying shear, with $u(t) = u_h \cos(2\pi t/T)$, and constant $\alpha$,

$$\left.\begin{aligned}
\langle x\rangle &= \beta \frac{u_h}{2\alpha}\left\{1 - e^{-2\alpha t}\left(\cos\frac{2\pi t}{T} - \frac{\pi}{\alpha T}\sin\frac{2\pi t}{T}\right)\right\}R \\
\langle x^2\rangle &= \frac{u_h}{4\alpha}\, t\left\{1 + \frac{T}{4\pi t}\sin\frac{4\pi t}{T} + \frac{1}{4\alpha t}\left(-\cos\frac{4\pi t}{T}\right)\right. \\
&\quad \left. - \frac{1}{\alpha t}\left[1 - e^{-2\alpha t}\left(\cos\frac{2\pi t}{T} - \frac{\pi}{\alpha T}\sin\frac{2\pi t}{T}\right)\right]R\right\}R \\
f(t) &= \frac{u_h^2}{2\alpha}\cos^2\frac{2\pi t}{T}\left\{1 + \frac{\pi}{\alpha T}\tan\frac{2\pi t}{T} - \sec\frac{2\pi t}{T}e^{-2\alpha t}\right\}R \\
R &= \frac{(\alpha T)}{(\alpha T) + \pi^2}
\end{aligned}\right\} . \tag{14}$$

Each expression contains a factor $R$, which indicates that high frequency shear does not contribute to the dispersion, since $R \to 0$ as $\alpha T \to 0$. This result was also found by Schönfeld (1961) for this model and by Okubo (1967) for another harmonic shear flow. The expressions that should be compared with time-dependent diffusion are those valid in the long time limit $\alpha t \gg 1$. With $R = 1$, these expressions are

$$\left.\begin{aligned}
\langle x\rangle &= \beta \frac{u_h}{2\alpha} \\
\langle x^2\rangle &= \frac{u_h^2}{4\alpha}\, t \\
f(t) &= \frac{u_h^2 \cos^2 \frac{2\pi t}{T}}{2\alpha} = \frac{[u(t)]^2}{2\alpha}
\end{aligned}\right\} . \tag{15}$$

Thus, low frequency harmonic shear can be parameterized as time-dependent diffusion with $K^* = [u(t)]^2/2\alpha$.

Schönfeld (1961), Bowden (1965), and Okubo (1967) have attempted to describe harmonic shear as time-independent diffusion. They obtained a diffusion coefficient given by $(1/T)\int_0^T K^*(t)\,dt$, the time average of $K^*(t)$ over one cycle of the shear. This yields the same result for $\langle x^2\rangle$ in the diffusion limit, but does not allow for generalization from harmonic shear to arbitrary shear.

**Case Three.** The question remains of how the diffusivity should be parameterized for the general case in which the shear and mixing vary arbitrarily in time. The answer should be obtainable by analyzing equation (12). This analysis is difficult because at any point in time there are three time scales that must be considered: the inverse of the mixing rate, $t_1=\alpha^{-1}$; the time over which the mixing varies, $t_2=\alpha/(d\alpha/dt)$; and the time over which the shear varies, $t_3=u/(du/dt)$. On the basis of the results for cases one and two and the discussion of Thacker (1976), it seems reasonable that the result of the analysis for the general case should be

$$K^* = \frac{[u(t)]^2}{2\alpha(t)} \tag{16}$$

for $t_2 \gg t_1$ and $t_3 \gg t_1$.

## 4. Discussion

The result presented here is that shear dispersion can be parameterized as enhanced diffusion along the shear even for time-dependent flows, so long as sufficient time has elapsed for the cross-shear mixing to take place. This is an extension of a previous time-independent analysis (Thacker, 1976). All of the conclusions from that analysis should also apply for the time-dependent case. First, the diffusion parameterization of the mixing should be used when the details of the shear are not resolved. This is equivalent to ignoring variations in times shorter than the time required to mix contaminant across the shear and to ignoring variations along the shear in distances shorter than the distance over which the shear stretches the contaminant in the cross-shear mixing time. Second, the effect of turbulent mixing can be accounted for simply by adding the turbulent eddy diffusivity to the diffusivity that parameterizes the shear dispersion. Finally, a more sophisticated description of the shear flow, such as that discussed by Bowden (1965), would lead to a more complicated, and perhaps more accurate, prescription for relating the effective diffusivity parameter to the shear flow. However, the result found here should also hold for more sophisticated descriptions of the shear flow. The time-dependent diffusivity can be obtained from the expression for the time-independent diffusivity simply by allowing the flow parameters in that expression to vary in time.

## 5. References

Bowden, K. F., 1965. Horizontal mixing in the sea due to a shearing current. *J. Fluid Mech.*, **21**:83–95.

Okubo, A., 1967. The effect of shear in an oscillatory current on horizontal diffusion from an instantaneous source. *Int. J. Oceanol. Limnol.*, **1**: 194–204.

Schönfeld, J. C., 1961. The mechanism of longitudinal diffusion in a tidal river. *Bull. Int. Assoc. Sci. Hydrol.* (Louvain, Belgium), **6**(1).

Thacker, W. C., 1976. A solvable model of "shear dispersion." *J. Phys. Oceanogr.*, **6**:66–75.

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