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Relationships Between Several Fourth-Order Velocity Statistics and the Pressure Structure Function for Isotropic, Incompressible Turbulence

Reginald J. Hill

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U.S. DEPARTMENT OF COMMERCE
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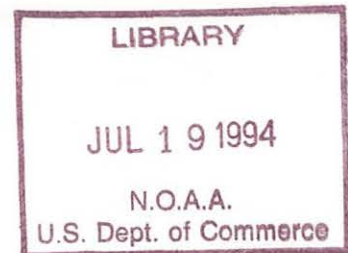


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Environmental Technology Laboratory
Boulder, Colorado

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Reginald J. Hill

ABSTRACT. Assumptions of isotropy, incompressibility, and joint Gaussian probability distribution for velocities at two points have previously been used to relate the pressure structure function to fourth-order velocity correlations. This relationship predicts that the pressure structure function varies as $r^{4/3}$ in the inertial range, and hence the pressure spectrum varies as $k^{-7/3}$. We show that the assumption of joint Gaussian velocities is implausible in this inertial-range application. We obtain a new theory relating the pressure structure function and spectrum to fourth-order velocity structure functions. We do not use the joint Gaussian assumption or any alternative approximation. The only assumptions are isotropy, incompressibility, and use of the Navier-Stokes equation. Specific formulas are given for pressure variance, mean-squared pressure gradient, and the viscous range of the pressure structure function. For the case of large Reynolds numbers, formulas are given for the inertial range of the pressure structure function and spectrum; these are valid on the less restrictive assumption of local isotropy, as are the formulas for mean-squared pressure gradient and the viscous range of the pressure structure function. Using the experimentally verified extension to fourth-order velocity structure functions of Kolmogorov's theory, for the inertial range of the pressure structure function and spectrum, we obtain $r^{4/3}$ and $k^{-7/3}$ laws. The modifications of these power laws to account for the effects of turbulence intermittency are also given. As a result of intermittency, a slightly smaller exponent than $4/3$ is obtained for the inertial range, whereas the joint Gaussian assumption leads to a slightly larger exponent. The inertial-range behavior of the fourth-order velocity correlation tensor and its corresponding structure function is investigated using the approximation that velocity components are statistically independent of the differences of velocity components, the latter being obtained at spatial separations that lie within the inertial range. This assumption is more plausible for the inertial range and less restrictive than the joint Gaussian assumption. This investigation reveals that in the inertial range some types of fourth-order structure functions of velocity are proportional both to $r^{2/3}$ and to velocity covariance. Another type of fourth-order structure function of velocity is proportional to $r^{4/3}$ and is not proportional to velocity covariance.

1. INTRODUCTION

Batchelor (1951), Obukhov (1949), and Obukhov and Yaglom (1951) used the assumption that velocities at two points are joint Gaussian random variables to derive formulas for the pressure structure function. We show that velocity differences are relevant to deducing the pressure structure function. The important aspect of the assumption of joint Gaussian two-point velocities is that the difference of velocities at two points is Gaussian. This can be an adequate assumption if the two points are separated by a distance lying within the variance-containing range. However, for separations in the inertial and viscous ranges this assumption is poor, as shown by Anselmet et al. (1984) and Van Atta and Park (1972). Consequently, the use of the joint Gaussian assumption and the resulting deductions are poorly founded. On the basis of the joint Gaussian assumption, Batchelor (1951) and Obukhov (1949) showed that the pressure structure function varies as $r^{4/3}$ within the inertial range; hence, the pressure spectrum varies as $k^{-7/3}$. Obukhov and Yaglom (1951) also established this power law on the basis of dimensional analysis. What are in doubt are the proportionality constant of the inertial-range power law; the mean-squared pressure gradient, which multiplies r^2 in the viscous-range formula; the transition between these ranges; the effects of turbulence intermittency; and the general formula relating the pressure structure function to velocity statistics.

Further discussion of the applicability of the Gaussian distribution is given by Van Atta and Wyngaard (1975). In this regard, Van Atta and Wyngaard (1975) and Batchelor (1951) noted that the joint Gaussian assumption cannot be precisely correct because it predicts zero odd-order moments. We, too, examine the implications of joint Gaussian velocity statistics, but with reservations as to its applicability. An alternative approximation for inertial and viscous ranges at high Reynolds number is the statistical independence of velocity from velocity differences. We use this alternative to derive the inertial-range dependence of fourth-order velocity statistics and compare these with the corresponding result from the joint Gaussian assumption. However, this statistical independence approximation is not used to derive the pressure structure function.

Use of the joint Gaussian assumption and incompressibility causes cancellation of the dominant terms contributing to fourth-order velocity correlations. We make this clear by expressing the result of the joint Gaussian assumption in terms of second-order velocity structure functions, rather than in terms of second-order correlations. The inertial range of the pressure structure function is then determined by relatively very small differences that contribute to fourth-order velocity correlations.

Without applying the joint Gaussian assumption, the fourth-order velocity correlations are not useful for predicting the inertial range of the pressure structure function. To see this, consider that even accurate measurements of fourth-order velocity correlations could not be used to accurately obtain the pressure structure function for the inertial range. An attempt by Uberoi (1953) failed to obtain the pressure structure function from measured fourth-order velocity correlations. We seek another fourth-order velocity statistic that is related to pressure fluctuations but that does not contain large terms that cancel when the incompressibility condition is imposed. The structure function consisting of the product of four differences of velocity components is the desired statistic. This statistic has the advantages that it has the simplest possible isotropic form and that one of its components has been extensively studied theoretically and experimentally. This allows us to derive a simple general formula relating the pressure structure function to integrals of components of this fourth-order velocity structure function, and to derive simple analytic formulas for asymptotic ranges. These are derived without replacing the joint Gaussian assumption with an alternative approximation. From these formulas, we obtain pressure variance, mean-squared pressure gradient, inertial and viscous range formulas, and the acceleration tensor. Much analytical manipulation is needed to obtain these results; it is summarized in Appendices A-K.

2. RELATIONSHIPS, DEFINITIONS, AND CONVENTIONS

The divergence of the Navier-Stokes equation gives the following relationship of the Laplacian of pressure to the velocity derivatives for incompressible fluid (Batchelor, 1951):

$$\frac{1}{\rho} \nabla^2 P = - (u_i u_j)_{|ij}, \quad (1)$$

where P is pressure, ρ is fluid density, u_i and u_j are velocity components, and ∇^2 is the Laplacian operator. We use the notation

$$(u_i u_j)_{|ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j} (u_i u_j). \quad (2)$$

Summation is implied over repeated Roman indices; no summation is implied for repeated Greek indices. Using (1), Batchelor (1951) and Obukhov and Yaglom (1951) derived the pressure structure function and correlation for isotropic turbulence in incompressible fluid; for the structure function, their result is

$$\begin{aligned} D_p(r) &\equiv \frac{1}{\rho^2} \langle (P - P')^2 \rangle \\ &= \frac{1}{3r} \int_0^r (y^4 - 3ry^3 + 3r^2y^2) Q(y) dy + \frac{r^2}{3} \int_r^\infty y Q(y) dy. \end{aligned} \quad (3)$$

Spatial positions are denoted by \vec{x} and \vec{x}' ; then, $\vec{r} \equiv \vec{x} - \vec{x}'$, $r \equiv |\vec{r}|$. Angle brackets denote averaging, and we use the convention

$$P \equiv P(\vec{x}) \quad \text{and} \quad P' \equiv P(\vec{x}'). \quad (4)$$

In (3), Q is defined by

$$Q(r) \equiv \langle (u_i u_j)_{|ij} (u'_k u'_l)_{|kl} \rangle \quad (5)$$

$$= \langle u_i u_j u'_k u'_l \rangle_{|ijkl}. \quad (6)$$

In (6), the differentiation is with respect to the components of \vec{r} .

The fourth-order velocity correlation is defined by

$$R_{ijkl}(\vec{r}) \equiv \langle u_i u_j u'_k u'_l \rangle. \quad (7)$$

Thus $Q(r)$ can be obtained by differentiation of $R_{ijkl}(\vec{r})$ and summation over indices. $Q(r)$ can also be obtained from other fourth-order statistics; we consider the following fourth-order structure functions:

$$\Delta_{ijkl}(\vec{r}) \equiv \left\langle (u_i u_j - u'_i u'_j) (u_k u_l - u'_k u'_l) \right\rangle, \quad (8)$$

and

$$D_{ijkl}(\vec{r}) \equiv \left\langle (u_i - u'_i) (u_j - u'_j) (u_k - u'_k) (u_l - u'_l) \right\rangle. \quad (9)$$

The relationships between these quantities are

$$\Delta_{ijkl}(\vec{r}) = 2 [R_{ijkl}(0) - R_{ijkl}(\vec{r})] \quad (10)$$

$$D_{ijkl}(\vec{r}) = 2 [R_{ijkl}(\vec{r}) + R_{ikjl}(\vec{r}) + R_{iljk}(\vec{r}) - 3R_{ijkl}(0)] + M_{ijkl}(\vec{r}) \quad (11)$$

$$= -\Delta_{ijkl}(\vec{r}) - \Delta_{ikjl}(\vec{r}) - \Delta_{iljk}(\vec{r}) + M_{ijkl}(\vec{r}), \quad (12)$$

where

$$M_{ijkl}(\vec{r}) = 2 \left[4R_{ijkl}(0) - \left(\langle u'_i u_j u_k u_l \rangle + \langle u_i u'_j u_k u_l \rangle + \langle u_i u_j u'_k u_l \rangle + \langle u_i u_j u_k u'_l \rangle \right) \right] \quad (13a)$$

$$= 2 \left[\langle (u_i - u'_i) u_j u_k u_l \rangle + \langle u_i (u_j - u'_j) u_k u_l \rangle + \langle u_i u_j (u_k - u'_k) u_l \rangle + \langle u_i u_j u_k (u_l - u'_l) \rangle \right]. \quad (13b)$$

For a correlation having one velocity component evaluated at a point different from the other components, the incompressibility condition gives, for instance,

$$\langle u'_i u_j u_k u_l \rangle_{|i} = 0. \quad (14)$$

Therefore, by differentiating (13a),

$$M_{ijkl}(\vec{r})_{|ijkl} = 0. \quad (15)$$

Thus (6), (7), (10), (11), and (15) give

$$R_{ijkl}(\vec{r})_{|ijkl} = Q(r), \quad (16)$$

$$\Delta_{ijkl}(\vec{r})_{|ijkl} = -2Q(r), \quad (17)$$

$$D_{ijkl}(\vec{r})_{|ijkl} = 6Q(r), \quad (18)$$

which shows three ways to generate $Q(r)$. For spacings in the inertial range, the relationship (10) expresses $\Delta_{ijkl}(\vec{r})$ as the difference of quantities that are much bigger than $\Delta_{ijkl}(\vec{r})$; likewise, (11) and (12) express $D_{ijkl}(\vec{r})$ as the difference of quantities much larger than itself. Yet, (16) to (18) show derivatives of similar magnitude.

We need to define the second-order velocity covariance

$$R_{ij}(\vec{r}) \equiv \langle u_i u_j' \rangle, \quad (19)$$

and for brevity and clarity, the covariance

$$\sigma_{ij} \equiv \langle u_i u_j \rangle = R_{ij}(0). \quad (20)$$

The corresponding structure function is

$$D_{ij}(\vec{r}) \equiv \langle (u_i - u_i')(u_j - u_j') \rangle = 2 [\sigma_{ij} - R_{ij}(\vec{r})]. \quad (21)$$

We have need of a special coordinate system; we will call this the chosen coordinate system. The chosen coordinate system is Cartesian with its 1-axis aligned along the separation vector \vec{r} . When we refer to specific components of the tensors, such as Δ_{1111} , D_{1111} , D_{11} , D_{22} , D_{33} , we imply that these components are taken along axes of the chosen coordinate system. Thus, we will not repeat mention of the chosen coordinate system when we present results or refer to a tensor's components. These components depend only on the spacing r , not on all components of \vec{r} separately. Greek indices are used to denote a general index for a component resolved in the chosen coordinate system [e.g., $\Delta_{\alpha\alpha\beta\beta}(r)$]. No summation is implied by repeated Greek indices. For isotropic turbulence, tensors have specific formulas in terms of scalar functions. These formulas are given in Appendix A for the tensors under consideration as well as for their derivatives.

In addition to our general results, we also consider the particular case of very large Reynolds numbers, such that the outer scale and microscale are sufficiently separated that an inertial range exists wherein (Kolmogorov, 1941)

$$D_{11}(r) = \frac{3}{4} D_{22}(r) = \frac{3}{4} D_{33}(r) = C \varepsilon^{2/3} r^{2/3}, \quad (22)$$

where ε is the energy dissipation rate and C is the Kolmogorov constant; Yaglom (1981) reviewed data and recommended $C \approx 2$. This relationship of $D_{11}(r)$ to $D_{22}(r)$ and $D_{33}(r)$ and the fact that $D_{\alpha\beta}(r) = 0$ for $\alpha \neq \beta$ is obtained from (A24) and (A25). We define an outer scale L_0 such that the inertial-range formula for $D_{11}(r)$ equals $D_{11}(\infty)$; that is,

$$C \varepsilon^{2/3} L_0^{2/3} = 2 \sigma_{11}. \quad (23)$$

For isotropic turbulence, $\sigma_{11} = \sigma_{22} = \sigma_{33}$. Then, (22) and (23) give

$$D_{11}(r)/\sigma_{11} = \frac{3}{4} D_{22}(r)/\sigma_{22} = \frac{3}{4} D_{33}(r)/\sigma_{33} = 2 (r/L_0)^{2/3}. \quad (24)$$

Mention of the words "asymptotic," "asymptotically," etc., refer to spacing r within the inertial range, taken sufficiently small that r/L_0 approaches zero; this requires asymptotically large Reynolds numbers. Thus, the asymptotic case always refers to the inertial range.

We investigate the small-scale range of $\Delta_{ijkl}(\vec{r})$ using the approximation that, in the asymptotic case, the averages of velocity components are statistically independent (*SI*) of differences of velocity components. Many nonlinear cascades result in intense local velocity differences, whereas the velocity varies mostly on the scale of the variance-containing range. Of course, this *SI* approximation is not exact, and we apply it only to velocity moments having an even number of velocity components evaluated at the same position. We also consider a variant of the *SI* approximation in which the velocity is assumed to be statistically independent of the velocity difference. The variant *SI* approximation seems to be inferior to the former *SI* approximation. For instance, the variant *SI* approximation wrongly predicts that M_{ijkl} is zero, because of (13b) and $\langle u_i - u'_i \rangle = 0$. Thus, this variant *SI* approximation is of no use for predicting M_{ijkl} or for balancing (12). We use both *SI* approximations in a very restricted manner. We emphasize that this *SI* approximation is not used to derive our formulas for the pressure structure function. The superscript *SI* indicates that a formula derives from this statistical independence approximation. This *SI* approximation was used by Lumley (1965) and Wyngaard and Clifford (1977) to obtain corrections to Taylor's frozen-flow hypothesis.

We will show formulas that follow from the assumption that the velocities at different points have a joint Gaussian probability distribution. This assumption relates fourth-order correlations and structure functions to those of second order. For brevity, we will call this the *JG* assumption. For clarity, the tensors will be labeled with the superscript *JG* [e.g., $D_{ijkl}^{JG}(\vec{r})$, $\Delta_{ijkl}^{JG}(\vec{r})$] when a formula derives from the *JG* assumption.

3. THE r -DEPENDENCE OF FOURTH-ORDER VELOCITY STRUCTURE FUNCTIONS

We first briefly discuss the r -dependence of the fourth-order structure functions at very large (production range) and very small (dissipation range) separations; then we discuss the inertial-range r -dependence in detail. For large separations, the following are readily obtained:

$$R_{ijkl}(\infty) = \sigma_{ij}\sigma_{kl} \quad (25a)$$

$$\Delta_{ijkl}(\infty) = 2[R_{ijkl}(0) - \sigma_{ij}\sigma_{kl}] \quad (25b)$$

$$M_{ijkl}(\infty) = 8R_{ijkl}(0) \quad (25c)$$

$$D_{ijkl}(\infty) = 2[R_{ijkl}(0) + \sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}]. \quad (25d)$$

For the moment, let no two of the indices α , γ , or λ be equal; from isotropy, we have the following relationships between nonvanishing components:

$$R_{\alpha\alpha\gamma\gamma}(0) = R_{\lambda\lambda\gamma\gamma}(0) \quad (26a)$$

$$R_{\alpha\alpha\alpha\alpha}(0) = R_{\lambda\lambda\lambda\lambda}(0). \quad (26b)$$

Note that, unlike $R_{ijkl}(\vec{r})$, $R_{ijkl}(0)$ is symmetric under interchange of any pair of indices. From (25d) and (26a,b), we have

$$D_{\alpha\alpha\gamma\gamma}(\infty) = D_{\lambda\lambda\gamma\gamma}(\infty) \quad (26c)$$

$$D_{\alpha\alpha\alpha\alpha}(\infty) = D_{\lambda\lambda\lambda\lambda}(\infty). \quad (26d)$$

With $D_{ijkl}(\infty)$ replaced by $M_{ijkl}(\infty)$ or $\Delta_{ijkl}(\infty)$, (26c,d) remain valid [caution: $\Delta_{ijkl}(\infty)$ is not completely symmetric]. From Table A1 or (A20) and (A21), we have, taking $r \rightarrow \infty$, for λ and γ taken to be 2 or 3 and $\gamma \neq \lambda$,

$$D_{\lambda\lambda\gamma}(\infty) = \frac{1}{3} D_{\lambda\lambda\lambda}(\infty). \quad (27a)$$

Using (27a) in (25d) gives

$$R_{\lambda\lambda\gamma}(0) = \frac{1}{3} R_{\lambda\lambda\lambda}(0). \quad (27b)$$

Combining (27a) with (26c) as well as (27b) with (26a), with $\alpha = 1$, shows that (27a,b) remain valid if $\gamma = 1$; then (26b,d) show that λ can be 1 if $\gamma \neq 1$. Thus, (27a,b) are valid if λ and γ are 1, 2, or 3, but $\lambda \neq \gamma$. Combining (27a) and (26c) with $\alpha = 1$ gives (for $\gamma \neq 1$ and λ any of 1, 2, or 3, but now we can let $\gamma = \lambda$ or $\gamma \neq \lambda$, as we choose, provided $\gamma \neq 1$),

$$3D_{11\gamma}(\infty) - D_{\lambda\lambda\lambda}(\infty) = 0. \quad (28)$$

The results (28), or equivalently (26c), are surprisingly important because they are essential for finite pressure variance and the convergence of certain integrals. One can easily show that (26c,d), (27a), and (28) remain valid if D_{ijkl} is replaced by M_{ijkl} .

For r within the viscous range and tending toward zero, Taylor series expansion of the velocity readily shows that components of $\Delta_{ijkl}(\vec{r})$ and $M_{ijkl}(\vec{r})$ are proportional to r^2 and that components of $D_{ijkl}(\vec{r})$ are proportional to r^4 . The coefficients of proportionality are, of course, the appropriate velocity-derivative moments and mixed moments of velocity and velocity derivatives.

The remaining asymptotic range to be studied is the inertial range. Van Atta and Wyngaard (1975) considered "higher order" velocity correlation functions and their spectra. They limited their study to one velocity component; this corresponds to setting $i = j = k = l = 1$. The structure function Δ_{1111} thus corresponds to the "second-order" correlation functions and spectra considered by Van Atta and Wyngaard (1975). They presented data showing that higher-order spectra of the longitudinal velocity component vary as $k^{-5/3}$ in the inertial range. This corresponds to Δ_{1111} being proportional to $r^{2/3}$.

Using dimensional analysis, Van Atta and Wyngaard (1975) found that higher-order velocity spectra vary as $k^{-5/3}$ in the inertial range, implying $\Delta_{1111} \propto r^{2/3}$. This dimensional analysis is not deterministic because the number of parameters is one more than the number of applicable dimensions. Thus, as they noted, their result suggests but does not prove the $k^{-5/3}$ law for higher-order spectra. They showed that viscous dissipation is an important term in the budget of higher-order velocity moments; this dissipation rate plays a central role in their dimensional analysis. It is clear that the budget of the more general moment, $R_{\alpha\beta\gamma\lambda}(0)$, also has an important viscous-dissipation term. Hence, if we repeat their dimensional analysis, we obtain

$$\Delta_{\alpha\beta\gamma\lambda}(r) \propto \varepsilon_2(\alpha, \beta, \gamma, \lambda) \varepsilon^{-1/3} r^{2/3}, \quad (29)$$

where $\varepsilon_2(\alpha, \beta, \gamma, \lambda)$ is the dissipation rate for the moment $\langle u_\alpha u_\beta u_\gamma u_\lambda \rangle$.

Van Atta and Wyngaard (1975) also used the *JG* assumption to deduce that all higher-order spectra vary as $k^{-5/3}$. For their spectrum of u_1^2 , their result corresponds to

$$\Delta_{1111}^{JG} = 4\sigma_{11} D_{11}(r) \propto \sigma_{11} \varepsilon^{-1/3} r^{2/3}, \quad (30)$$

in agreement with a $k^{-5/3}$ spectrum. The dissipation rate in (29) is not necessarily proportional to the velocity variance in (30). Consequently, in (29) and (30), Van Atta and Wyngaard (1975) proposed two mutually exclusive theories. Given that the *JG* approximation is dubious when applied to the inertial range, it may seem that (29) is the more plausible theory. We think not. By rearranging terms in the definition (8) of Δ_{ijkl} and applying the *SI* approximation, we find that most nonvanishing components of Δ_{ijkl} are proportional to velocity variances. For the asymptotic (inertial range) case, the *SI* approximation is more plausible and less restrictive than the *JG* assumption, yet *SI* supports the *JG* predictions.

We now apply the *SI* approximation to Δ_{ijkl} . Table A1 shows that the nonvanishing components of Δ_{ijkl} in the chosen coordinate system are either of the type having $i = k$ and $j = l$, or $i = j$ and $k = l$; the case $i = j = k = l$ is a special case of these two types. The first type of nonvanishing component is

$$\begin{aligned} \Delta_{\alpha\beta\alpha\beta}(r) &= \langle [u_\alpha u_\beta - u'_\alpha u'_\beta]^2 \rangle \\ &= \left\langle \left[\frac{1}{2} (u_\beta + u'_\beta) (u_\alpha - u'_\alpha) + \frac{1}{2} (u_\alpha + u'_\alpha) (u_\beta - u'_\beta) \right]^2 \right\rangle \end{aligned} \quad (31a)$$

$$\begin{aligned} &= \langle u_\beta^2 (u_\alpha - u'_\alpha)^2 \rangle + \langle u_\alpha^2 (u_\beta - u'_\beta)^2 \rangle \\ &+ 2 \langle u_\beta u_\alpha (u_\alpha - u'_\alpha) (u_\beta - u'_\beta) \rangle - \langle (u_\alpha - u'_\alpha)^2 (u_\beta - u'_\beta)^2 \rangle. \end{aligned} \quad (31b)$$

Applying the *SI* approximation to (31b) gives

$$\begin{aligned} \Delta_{\alpha\beta\alpha\beta}^{SI}(r) &= \sigma_{\beta\beta} D_{\alpha\alpha}(r) + \sigma_{\alpha\alpha} D_{\beta\beta}(r) + 2\sigma_{\alpha\beta} D_{\alpha\beta}(r) \\ &- D_{\alpha\alpha\beta\beta}(r). \end{aligned} \quad (32)$$

The second type of nonvanishing component is

$$\begin{aligned}\Delta_{\alpha\alpha\beta\beta}(r) &= \left\langle \left(u_\alpha^2 - u_\alpha'^2 \right) \left(u_\beta^2 - u_\beta'^2 \right) \right\rangle \\ &= \left\langle \left(u_\alpha - u_\alpha' \right) \left(u_\alpha + u_\alpha' \right) \left(u_\beta - u_\beta' \right) \left(u_\beta + u_\beta' \right) \right\rangle\end{aligned}\quad (33a)$$

$$\begin{aligned}&= 4 \left\langle u_\alpha u_\beta \left(u_\alpha - u_\alpha' \right) \left(u_\beta - u_\beta' \right) \right\rangle \\ &\quad - \left\langle \left(u_\alpha - u_\alpha' \right)^2 \left(u_\beta - u_\beta' \right)^2 \right\rangle.\end{aligned}\quad (33b)$$

Applying our *SI* approximation to (33b) gives

$$\Delta_{\alpha\alpha\beta\beta}^{SI}(r) = 4 \sigma_{\alpha\beta} D_{\alpha\beta}(r) - D_{\alpha\alpha\beta\beta}(r). \quad (34)$$

For $\alpha = \beta$, both (32) and (34) give

$$\Delta_{\alpha\alpha\alpha\alpha}^{SI}(r) = 4 \sigma_{\alpha\alpha} D_{\alpha\alpha}(r) - D_{\alpha\alpha\alpha\alpha}(r). \quad (35)$$

Also, (32) and (34) give

$$\Delta_{\alpha\beta\alpha\beta}^{SI}(r) = \sigma_{\beta\beta} D_{\alpha\alpha}(r) + \sigma_{\alpha\alpha} D_{\beta\beta}(r) - D_{\alpha\alpha\beta\beta}(r) \quad \text{for } \alpha \neq \beta \quad (36a)$$

$$\Delta_{\alpha\alpha\beta\beta}^{SI}(r) = - D_{\alpha\alpha\beta\beta}(r) \quad \text{for } \alpha \neq \beta. \quad (36b)$$

These formulas can be summarized as

$$\Delta_{ijkl}^{SI}(\vec{r}) = \sigma_{ik} D_{jl}(\vec{r}) + \sigma_{jl} D_{ik}(\vec{r}) + \sigma_{il} D_{jk}(\vec{r}) + \sigma_{jk} D_{il}(\vec{r}) - D_{ijkl}(\vec{r}). \quad (37a)$$

If we use (22), (35) and (36a,b) give $r^{2/3}$ as the dominant asymptotic variation. If we use (24) and results for $D_{\alpha\alpha\beta\beta}(r)$ and $D_{\alpha\alpha\alpha\alpha}(r)$ in the following, $\Delta_{\alpha\beta\alpha\beta}(r)$ and $\Delta_{\alpha\alpha\alpha\alpha}(r)$ are asymptotically very much larger than $D_{\alpha\alpha\beta\beta}(r)$ and $D_{\alpha\alpha\alpha\alpha}(r)$. The results (35) and (36a,b) are obtained by assuming that u_α is statistically independent of velocity differences; the identical results are derived if u_β' is assumed to be statistically independent of velocity differences. An equally plausible approximation is that the average $(u_\alpha + u_\alpha')/2$ is statistically independent of velocity differences. Applying this to (31a) and (33a), we obtain that in (35), $D_{\alpha\alpha\alpha\alpha}(r)$ is replaced by $[D_{\alpha\alpha}(r)]^2$; in (36a), $D_{\alpha\alpha\beta\beta}(r)$ is replaced by

$D_{\alpha\alpha}(r) D_{\beta\beta}(r)/2$; and in (36b), $D_{\alpha\alpha\beta\beta}(r)$ is replaced by zero. Note that none of these replacements are the *JG* results for $D_{\alpha\alpha\beta\beta}^{JG}$ in (B15) and (B16). Thus, (37a) can equally plausibly be replaced by

$$\begin{aligned} \Delta_{ijkl}^{SI}(\vec{r}) &= \sigma_{ik} D_{jl}(\vec{r}) + \sigma_{jl} D_{ik}(\vec{r}) + \sigma_{il} D_{jk}(\vec{r}) + \sigma_{jk} D_{il}(\vec{r}) \\ &\quad - \frac{1}{2} [D_{ik}(\vec{r}) D_{jl}(\vec{r}) + D_{il}(\vec{r}) D_{jk}(\vec{r})]. \end{aligned} \quad (37b)$$

Therefore, we cannot justify retaining the last term in (35), (36a), and (37a,b) because these last terms may be smaller or on the order of the error of our approximation; (36b) may be on the order of the error of our approximation.

Now consider the implications of the joint Gaussian (*JG*) assumption. The manipulations are given in Appendix B; the results are

$$\begin{aligned} \Delta_{ijkl}^{JG}(\vec{r}) &= \sigma_{ik} D_{jl}(\vec{r}) + \sigma_{jl} D_{ik}(\vec{r}) + \sigma_{il} D_{jk}(\vec{r}) + \sigma_{jk} D_{il}(\vec{r}) \\ &\quad - \frac{1}{2} [D_{ik}(\vec{r}) D_{jl}(\vec{r}) + D_{il}(\vec{r}) D_{jk}(\vec{r})]. \end{aligned} \quad (38)$$

We note that (38) is the same as (37b).

In Appendix B, we obtain from (38) the components in our chosen coordinate system that are listed as nonvanishing in Table A1:

$$\Delta_{\alpha\alpha\alpha\alpha}^{JG}(r) = 4 \sigma_{\alpha\alpha} D_{\alpha\alpha}(r) - [D_{\alpha\alpha}(r)]^2 \quad (39)$$

$$\begin{aligned} \Delta_{\alpha\beta\alpha\beta}^{JG}(r) &= \sigma_{\alpha\alpha} D_{\beta\beta}(r) + \sigma_{\beta\beta} D_{\alpha\alpha}(r) \\ &\quad - \frac{1}{2} D_{\alpha\alpha}(r) D_{\beta\beta}(r) \quad \text{for } \alpha \neq \beta \end{aligned} \quad (40)$$

$$\Delta_{\alpha\alpha\beta\beta}^{JG}(r) = 0 \quad \text{for } \alpha \neq \beta. \quad (41)$$

For either approximation, components of $\Delta_{ijkl}(\vec{r})$ are either zero or vary asymptotically as $r^{2\beta}$, with the exception of (36b). We caution the reader that the *SI* approximation is asymptotic. If valid, it applies for very small r/L_0 , whereas the *JG* approximation, if valid, is most accurate for r larger than or on the order of L_0 . Given this caution, note that (37b) is identical to (38) and therefore (37b) gives (39) to (41), whereas (37a) differs from both (37b) and (38) in the asymptotically small terms. That equally plausible applications of *SI* produce different asymptotically small terms suggests that the

SI approximation is incapable of correctly obtaining these small terms, and this calls into question the plausibility of the correctness of these terms in the *JG* result (38). On the other hand, if one can show that (37b) is the proper *SI* result to the exclusion of (37a), then the asymptotically small terms in (38) become plausibly correct. These distinctions are important because these asymptotically small terms produce the pressure structure function.

We have investigated the predictions of both the *SI* and *JG* approximations. We conclude that most components of Δ_{ijkl} are proportional to velocity covariances and behave as $r^{2/3}$ in the inertial range. We believe that this is simply a consequence of being able to rewrite $\Delta_{\alpha\beta\alpha\beta}$ and $\Delta_{\alpha\alpha\beta\beta}$ in the forms (31a,b) and (33a,b) that make it evident that asymptotically there is a factoring into variance and second-order structure function.

We now turn to the inertial range of D_{ijkl} . The nonzero components of D_{ijkl} for isotropic turbulence are given in Table A1. Considering the symmetry of D_{ijkl} under exchange of indices, these nonzero components are all of the type having $i = j$ and $k = l$, the case $i = j = k = l$ being a special case of this same type. Therefore, we consider the component

$$D_{\alpha\alpha\beta\beta}(r) = \left\langle (u_\alpha - u'_\alpha)^2 (u_\beta - u'_\beta)^2 \right\rangle, \quad (42)$$

which we must understand for later application to the pressure structure function. Asymptotically, an appropriate scale for $(u_\alpha - u'_\alpha)^2$ is $D_{\alpha\alpha}(r)$. Thus, simple scaling considerations lead us to the estimate, for $\alpha \neq \beta$,

$$D_{\alpha\alpha\beta\beta}(r) \propto a D_{\alpha\alpha}(r) D_{\beta\beta}(r) \propto r^{4/3}, \quad (43)$$

and for $\alpha = \beta$,

$$D_{\alpha\alpha\alpha\alpha}(r) \approx b [D_{\alpha\alpha}(r)]^2 \propto r^{4/3}, \quad (44)$$

where a and b are unknown coefficients. Asymptotically, the estimates (43) and (44) must apply equally well over an arbitrarily large range of r . Hence, these estimates prescribe the dominant variation to be $r^{4/3}$.

The *JG* assumption leads to similar results. From Appendix B, we have

$$D_{ijkl}^{JG}(\vec{r}) = D_{ik}(\vec{r}) D_{jl}(\vec{r}) + D_{il}(\vec{r}) D_{jk}(\vec{r}) + D_{ij}(\vec{r}) D_{kl}(\vec{r}). \quad (45)$$

The nonvanishing components in our chosen coordinate system are

$$D_{\alpha\alpha\alpha\alpha}^{JG}(r) = 3 [D_{\alpha\alpha}(r)]^2 \quad (46)$$

$$D_{\alpha\alpha\beta\beta}^{JG}(r) = D_{\alpha\alpha}(r) D_{\beta\beta}(r) \quad \text{for } \alpha \neq \beta. \quad (47)$$

Note the similarity of (43) and (47) and of (44) and (46). For an inertial range, we can obtain similar results from dimensional analysis using ε and r as relevant parameters, but this brings us to the following discussion of intermittency theory.

The velocity structure functions given by $\langle (u_1 - u'_1)^n \rangle$ have been studied extensively in connection with the effects of intermittency. The relevance to the pressure structure function arises from

$$D_{1111}(r) \equiv \langle (u_1 - u'_1)^4 \rangle. \quad (48)$$

The experimental studies of (48) were by Anselmet et al. (1984), Antonia et al. (1982a), Vasilenko et al. (1975), Van Atta and Park (1972), and Van Atta and Chen (1970). They demonstrated that $(u_1 - u'_1)$ is not a Gaussian random variable for spacings in the inertial range, that the flatness factor $\langle (u_1 - u'_1)^4 \rangle / \langle (u_1 - u'_1)^2 \rangle^2$ is not 3 as implied by (46) (it is larger and varies with the Reynolds number), that the variation of $D_{1111}(r)$ is slightly less steep than $r^{4/3}$, and that the flatness factor varies somewhat with r .

The various theories of the intermittency effect are negligibly different when applied to $D_{1111}(r)$. These theories were reviewed by Anselmet et al. (1984). Here we state the prediction of the earliest theory as given by Kolmogorov (1962):

$$D_{1111}(r) = C_{11} \varepsilon^{4/3} r^{4/3 - 2\mu/9}, \quad (49)$$

where C_{11} depends on the flow macrostructure. Indeed, C_{11} has dimensions of a length raised to the power of $2\mu/9$. Experiments give $\mu = 0.25 \pm 0.05$ (Sreenivasan and Kailasnath, 1993), so $2\mu/9 = 0.06$, which gives a very small departure from the 4/3 power law. The intermittency theories and experiments show that the simple scaling consideration that gave (43) and (44) cannot be extended to high-order moments.

We can now state the asymptotic inertial-range formula for M_{ijkl} . We use the fact that M_{ijkl} must exactly cancel the $r^{2/3}$ asymptotic behavior of the Δ_{ijkl} in (12) such that D_{ijkl} behaves as $r^{4/3}$ asymptotically. Note that M_{ijkl} has the same symmetry under interchange of its indices as does D_{ijkl} ; thus, for instance, $M_{\alpha\beta\alpha\beta} = M_{\alpha\alpha\beta\beta}$. From (12), the asymptotically dominant, nonvanishing components in our chosen coordinate system are

$$M_{\alpha\alpha\alpha\alpha}(r) \approx 3\Delta_{\alpha\alpha\alpha\alpha}(r) \quad (50)$$

$$M_{\alpha\alpha\alpha\alpha}^{SI}(r) = M_{\alpha\alpha\alpha\alpha}^{JG}(r) \approx 12\sigma_{\alpha\alpha}D_{\alpha\alpha}(r) \quad (51)$$

$$M_{\alpha\alpha\beta\beta}(r) \approx \Delta_{\alpha\alpha\beta\beta}(r) + 2\Delta_{\alpha\beta\alpha\beta}(r) \quad \text{for } \alpha \neq \beta \quad (52)$$

$$M_{\alpha\alpha\beta\beta}^{SI}(r) = M_{\alpha\alpha\beta\beta}^{JG}(r) \approx 2\sigma_{\alpha\alpha}D_{\beta\beta}(r) + 2\sigma_{\beta\beta}D_{\alpha\alpha}(r) \quad \text{for } \alpha \neq \beta. \quad (53)$$

The expressions (51) and (53) follow respectively from (50) and (52) using either *JG* or *SI* approximations in (35) to (36a,b) or (39) to (41); (51) and (53) demonstrate that components of M_{ijkl} are asymptotically proportional to velocity covariance and to $\epsilon^{-1/3}r^{2/3}$.

4. ASYMPTOTIC FORMULAS FOR THE PRESSURE STRUCTURE FUNCTION

We discuss the asymptotic formulas for $D_p(r)$ for the inertial, dissipation, and production ranges. At sufficiently large separations in the production range, $D_p(r) \approx D_p(\infty)$, which is twice the pressure variance. The pressure variance is discussed in Appendix H. To obtain $D_p(r)$ in the dissipation range, we first use the formula by Batchelor (1951):

$$\chi \equiv \frac{1}{\rho^2} \langle |\nabla P|^2 \rangle = \int_0^\infty y Q(y) dy. \quad (54)$$

We see that χ is the mean-squared pressure gradient, which is discussed in Appendix G. Using (54), we replace the last integral in (3) with an integral from 0 to r . Then, by Taylor series expansion of $Q(y)$, (3) gives as $r \rightarrow 0$,

$$D_p(r) = \frac{1}{3}\chi r^2 - \frac{1}{60}Q(0)r^4 + \dots, \quad (55)$$

where

$$Q(0)/60 = d_{11}h_Q,$$

$$h_Q \equiv 1 + \frac{1}{3}\frac{d_{\lambda\lambda}}{d_{11}} - 3\frac{d_{1\gamma}}{d_{11}},$$

and the derivative moments d_{11} , $d_{\lambda\lambda}$, and $d_{1\gamma}$ are defined in Appendix F. We see that h_Q is a universal constant. Thus, $D_p(r)$ is quadratic at the origin, as is required by Taylor series expansion of the pressure. Of course, the *JG* approximation produces the same result with the additional, unlikely simplification that

$$Q^{JG}(0)/60 = \left\langle \left(\frac{du_i}{dx_i} \right)^2 \right\rangle. \quad (56)$$

We now investigate the inertial range of $D_p(r)$. Obukhov (1949) and Batchelor (1951) performed the derivative in (16) on the formula (B1) relating $R_{ijkl}(\vec{r})$ to the second-order velocity correlations. This is equivalent to using (17) and (38), since $R_{ijkl}(\vec{r})$ differs from $\Delta_{ijkl}(\vec{r})$ by an additive constant and a factor of -2 . Superficially, fourfold differentiation of a function that varies as $r^{2/3}$, as does (B1) and (38), will produce $Q(r) \propto r^{2/3-4}$, which, when substituted in (3), gives $D_p(r) \propto r^{2/3}$. However, the incompressibility condition is (Batchelor, 1960)

$$D_{ij}(\vec{r})|_i = -2R_{ij}(\vec{r})|_i = 0 \quad (57)$$

and

$$D_{ij}(\vec{r})|_j = -2R_{ij}(\vec{r})|_j = 0. \quad (58)$$

Hence, performing the differentiation in (16) or (17) causes the asymptotically dominant terms to vanish; this is most obvious from (38), (57), and (58). Only the asymptotically small terms in (38) produce $Q(r) \propto r^{4/3-4}$ and, hence, $D_p(r) \propto r^{4/3}$. We would be very fortunate if the JG assumption is so accurate that the asymptotically small terms in (38) were correct. If we instead generate $Q(r)$ using (18) and (45), then we obtain $Q(r) \propto r^{4/3-4}$. Only terms proportional to $r^{4/3}$ exist in (45). Indeed, we obtain the same $Q(r)$ from (18) and (45) as we obtain from (17) and (38), without the vanishing of asymptotically dominant terms.

The SI approximation is important in validating the inertial-range formulas (35) and (36a); it is, however, demonstrably incapable of predicting the inertial range of $D_p(r)$. To see this, note that (37) gives $\Delta_{ijkl}^{SI}(\vec{r})|_{ijkl} = -D_{ijkl}(\vec{r})|_{ijkl}$, which, by (17) and (18), is wrong by a factor of 3. Thus, the asymptotically small terms in (35), (36a), and perhaps (36b), are indeed in error as we suspected earlier.

Generating $Q(r)$ using (18) is therefore far more convincing than using (16) or (17), but we have noted that the JG assumption is perilously in disagreement with the experiment, so we do not want to use the JG assumption. We have the empirical result that $D_{1111}(r)$ is very nearly proportional to $r^{4/3}$, but we do not have an empirical basis for the other components of $D_{ijkl}(\vec{r})$ that are required in (18). The estimates (43) and (44) suggest that the other components are also proportional to $r^{4/3}$. Equation (A20) shows that $D_{1111}(r)$ is a linear combination of all three scalar functions that enter into the isotropic-tensor formula for $D_{ijkl}(\vec{r})$. Thus, if one of the other nonvanishing components [e.g., $D_{\alpha\alpha\beta\beta}(r)$, for $\alpha \neq \beta$] were to decrease more slowly than $r^{4/3}$, e.g., as $r^{2/3}$, then at some values of r , however small, $D_{1111}(r)$ would also have this same asymptotic variation. This is obvious from (A21) because at least one of the scalar functions must have such slower variation. Moreover, a transition within the inertial range to a decrease more slowly than (49) (essentially $r^{4/3}$) would

contradict the dimensional analysis leading to (49), and would also contradict the dimensional analysis by Obukhov and Yaglom (1951) that gives $D_p(r) \propto r^{4/3}$. Since this transition to a more gentle decrease has not been observed by measurements of $D_{1111}(r)$, we conclude either that the other components decrease at least as rapidly as $r^{4/3}$ or that the experiments have not yet attained sufficiently large Reynolds numbers for the slower variation to be observed. To determine which of these two possibilities is true, it would be best to measure all the structure-function components.

We assume that all the components of $D_{ijkl}(\vec{r})$ decrease at least as rapidly as $r^{4/3}$. Then, from (A22) and (18), the asymptotic dependence of $Q(r)$ is proportional to $r^{4/3-4}$. Hence, $D_p(r) \propto r^{4/3}$, and the pressure spectrum varies as $k^{-7/3}$. From (A22), the proportionality factor depends on the levels of three structure-function components, two of which have not been measured. From our previous comments, it would be fortuitous if the proportionality factor were the same as predicted on the basis of the *JG* assumption by Obukhov (1949) and Batchelor (1951).

We make the stronger assumption that all the nonvanishing components of D_{ijkl} have the same power law in the inertial range. In Appendix C, we show that this leads to

$$D_p(r) = H_p C_{11} \epsilon^{4/3} r^{4/3-2\mu/9} \quad (59a)$$

$$= H_p D_{1111}(r), \quad (59b)$$

where H_p is a universal constant defined in (C13) and is to be determined by experiment, and (59b) follows from (59a) by use of (49). If we neglect intermittency effects (take $\mu = 0$), then C_{11} is a universal constant rather than having macrostructure dependence, and we define a new universal constant $C_p \equiv H_p C_{11}$; we obtain the simpler results

$$D_p(r) \approx \frac{5}{3} D_{1111}(r) + 3 D_{\lambda\lambda\lambda\lambda}(r) - 15 D_{11\gamma\gamma}(r) \quad (60)$$

$$\approx C_p \epsilon^{4/3} r^{4/3}. \quad (61)$$

In (60), λ and γ are 2 or 3 (one can take $\lambda = \gamma$), and $3 D_{2233}(r)$ can be substituted for $D_{\lambda\lambda\lambda\lambda}(r)$. Since $D_p(r) > 0$, (60) gives a bound on the relative values of the structure-function components. This bound does not derive from kinematics alone; it results from use of the Navier-Stokes equation. A less stringent bound from $\chi > 0$ is given in Appendix G.

5. GENERAL FORMULA FOR THE PRESSURE STRUCTURE FUNCTION

Our general formulation for the pressure structure function consists of substituting (A22) in (18) and the result into (3). Integration by parts reduces the general formulation to a useful and simple result. The details are given in Appendix D. We obtain from (D17)

$$D_p(r) = -\frac{1}{3} D_{1111}(r) + \frac{4}{3} r^2 \int_r^\infty y^{-3} [D_{1111}(y) + D_{\lambda\lambda\lambda}(y) - 6 D_{11\gamma\gamma}(y)] dy \quad (62)$$

$$+ \frac{4}{3} \int_0^r y^{-1} [D_{\lambda\lambda\lambda}(y) - 3 D_{11\gamma\gamma}(y)] dy,$$

where, as discussed below (A22), indices γ and λ can be taken to be either 2 or 3 and $3 D_{2233}$ can be substituted for $D_{\lambda\lambda\lambda}$. The inertial-range formulas (59a) and (60) can be obtained from (62).

The relationship between a structure function and its spectrum from data along a line can be written as (Tatarskii, 1971)

$$\Psi_p(k_1) = \frac{1}{\pi k_1} \int_0^\infty dr \sin(k_1 r) D_p^{(1)}(r), \quad (63)$$

where $D_p^{(1)}(r) = dD_p/dr$, and $\Psi_p(k_1)$ is normalized to give the pressure variance as in (C15). Inserting (62) in (63) and integrating by parts gives

$$\Psi_p(k_1) = \frac{1}{\pi} \left\{ \int_0^\infty dr \frac{\sin(k_1 r)}{(k_1 r)} N_D(r) + \frac{8}{3} \int_0^\infty dr \left[\frac{\sin(k_1 r)}{(k_1 r)^3} - \frac{\cos(k_1 r)}{(k_1 r)^2} \right] A_D(r) \right\}, \quad (64)$$

where

$$N_D(r) \equiv -\frac{r}{3} D_{1111}^{(1)}(r) - \frac{4}{3} D_{1111}(r) + 4 D_{11\gamma\gamma}(r),$$

and

$$A_D(r) = D_{1111}(r) + D_{\lambda\lambda\lambda}(r) - 6 D_{11\gamma\gamma}(r),$$

and where $A_D(r)$ is the same function that appears in (A14) and (A20). Note that an inertial-range formula is not to be substituted into (63) or (64); convergence of the integrals requires (C14).

A fast Fourier transform (FFT) on a time series of measured pressure, and use of Taylor's hypothesis, can give $\Psi_p(k_1)$. However, the analogous operation for the right side of (64) is to calculate the fourth-order velocity structure functions, calculate $N_D(r)$ and $A_D(r)$, and then perform the integrals in (64). There is no FFT that produces the spectra corresponding to the integrals on the right side of (64). Detrending and windowing are usually performed prior to an FFT. Using (64) to compare a measured pressure spectrum with measured velocity structure functions presents the problem of how to analyze the velocity data to produce a right side of (64) that corresponds to detrended and windowed pressure data.

6. SENSITIVITY TO DEPARTURES FROM ISOTROPY

George et al. (1984) measured the pressure spectrum in the mixing layer of an axisymmetric jet and compared it with theory. They showed that their pressure spectrum is caused by three terms that they call (1) 2nd-moment turbulence-shear interaction, (2) 3rd-moment turbulence-shear interaction, and (3) turbulence-turbulence interaction. They used the *JG* assumption to deduce their spectrum for the turbulence-turbulence interaction. Our $D_p(r)$ is caused by only this third interaction. Judging by comparison of theoretical and measured spectra in their Fig. 15, $D_p(r)$ in the inertial range is less than twice $D_p^{JG}(r)$, i.e., $D_p(r) < 2 [D_{11}(r)]^2$.

Antonia et al. (1982a) obtained the inertial-range flatness factor $D_{1111}(r)/[D_{11}(r)]^2 \approx 4.5$; this was obtained from an axisymmetric jet having nearly the same Reynolds number (based on nozzle diameter d and exit velocity) as existed in the experiment by George et al. (1984), but Antonia et al. (1982a) used the downstream position $50 d$ and George et al. (1984) used $1.5 d$ and $3.0 d$. Therefore, we assume that the flatness factor was at least 4.5 in the experiment by George et al. (1984). Using this estimate of the flatness factor and the observation from the George et al. data that $D_p(r) < 2 [D_{11}(r)]^2$, we obtain $D_p(r) < (5/3) D_{1111}(r)/4$ in the inertial range. Therefore, the three terms in (60) cancel each other to produce a $D_p(r)$ that is at least 4 times smaller than the first term in (60) and may be yet much smaller than the term of largest magnitude in (60), whichever term that may be.

Isotropy gives six ways to calculate (60). Two possible choices for λ , as well as for γ , give four ways. Replacing $D_{\lambda\lambda\lambda}(r)$ with its isotropic equivalent, $3 D_{2233}(r)$, and the two possible choices for γ give two more ways. All six ways to calculate (60) from a given velocity data set and given r within the inertial range must yield the same $D_p(r)$ to within some plausible error; if the data do so, then we say that the data are sufficiently isotropic for prediction of the inertial range of $D_p(r)$.

The *JG* assumption also produces an inertial-range formula that is sensitive to insufficient isotropy. This is not apparent in previous work by Obukhov (1949) and Batchelor (1951) because of the order in which relationships were used to simplify the result. We apply

the *JG* assumption while delaying use of (A25), which is derived on the basis of both incompressibility and isotropy, as well as delaying use of (22), which follows from (A25) on the basis of Kolmogorov's 2/3 power law. Applying the *JG* assumption to $R_{ijkl}(\vec{r})$, $\Delta_{ijkl}(\vec{r})$, and $D_{ijkl}(\vec{r})$ gives (B2), (B7), and (B8), respectively, which, when substituted in (16) to (18), give the same $Q^{JG}(r)$ in all three cases; $Q^{JG}(r)$ is given in (B23). Substituting this $Q^{JG}(r)$ in (3) and integrating by parts gives (E8), which, using (E9), is the same as applying the *JG* assumption to (62). Now assume that $D_{\alpha\alpha}(r) \propto r^{2/3}$ for $\alpha = 1, 2, 3$; substituting this in (E8) gives

$$D_p^{JG}(r) = 5 [D_{11}(r)]^2 + 9 [D_{\lambda\lambda}(r)]^2 - 15 D_{11}(r) D_{\gamma\gamma}(r). \quad (65)$$

Using (B15) and (B16), we find that the three terms above are just the *JG* estimates for the corresponding three terms in our (60). To this point, we have used isotropy, incompressibility, and the inertial-range power law. To estimate the three terms in (65) and to pass to the usual final *JG* result, we use (A25), which is now the same as using (22); we obtain

$$D_p^{JG}(r) = 5 [D_{11}(r)]^2 + 16 [D_{11}(r)]^2 - 20 [D_{11}(r)]^2 \quad (66)$$

$$= [D_{11}(r)]^2, \quad (67)$$

where the three terms in (66) correspond to those in (65). The terms in (66), and therefore the terms in (65), must be much more accurate than 1 in 20 to produce (67). Given velocity data, there are four ways to calculate $D_p^{JG}(r)$ from (65). These ways correspond to two choices for each of λ and γ ; (67) is a fifth way. All five ways must give nearly the same value of $D_p^{JG}(r)$; otherwise, the data are insufficiently isotropic to predict $D_p^{JG}(r)$. Equivalently, the data must satisfy (A25) and (22) within a corresponding accuracy such that $D_p^{JG}(r)$ is adequately predicted in the inertial range. If data contain only u_1 , so that other components of D_{ij} cannot be calculated, then these data cannot plausibly predict $D_p^{JG}(r)$ using (67) nor prove (67).

We have previously noted reservations as to the accuracy of the *JG* assumption when it is applied to inertial-range velocity statistics. To obtain $D_p^{JG}(r)$, we must require that the *JG* assumption be sufficiently accurate in its prediction of the three terms in (65) so that these terms cancel to accurately produce the much smaller result (67). We consider this level of accuracy to be implausible; it has not been demonstrated experimentally or theoretically.

We note that even if a flow is sufficiently isotropic, or sufficiently locally isotropic at some r , velocity data can be insufficiently isotropic because of imperfections in the measurement process (Karyakin et al., 1991). For instance, when the energy-containing range is anisotropic, use of Taylor's hypothesis can result in measured local anisotropy even if the flow has accurate local isotropy (Hill, 1994). Care must therefore be exercised when using Taylor's hypothesis to obtain either fourth-order velocity structure functions for use in (59a,b) and (60) or second-order structure functions for use in (65).

The formulas for the viscous range of $D_p(r)$ and the mean-squared pressure gradient are not as sensitive to the accuracy of isotropy as is the inertial-range formula.

The relationships (59a,b) and (65) are also sensitive to the value of the exponent in the inertial-range formulas. To illustrate this fact, let the exponent in the inertial-range formula for $D_{\alpha\alpha}(r)$ be $(2/3) + (\mu/9)$; this exponent must be the same for $\alpha = 1, 2,$ and 3 if (A25) is exact. We can let μ be any value, including the intermittency value. Using this exponent in (E8), to obtain a modification of (65), as well as in (A25), to further obtain a modified (22), we obtain the result that (67) is to be multiplied by $1 + (\mu/2)$. The intermittency value, $\mu \approx 0.25$, makes a perhaps unmeasurable change in second-order velocity structure functions and corresponding spectra, but it makes a 12% increase in the inertial range of $D_p^{JG}(r)$. A similar sensitivity is likely for (59a,b), but quantitative results must await measurements so that H_p can be evaluated.

7. EFFECTS OF COMPRESSIBILITY

Incompressibility is a very important assumption in our derivations; it produced great simplification relative to the case of compressible fluid flow. To derive the pressure structure function and pressure spectrum for the case of compressibility requires that we use the hydrodynamics equations for the compressible case. The derivation would be much more complicated than that presented in this paper.

However, we can make some inferences as to the importance of compressibility to the form of the pressure structure function. If $D_{ij|i}$ and $D_{ij|j}$ do not vanish, then it is plausible that (18) and (32) and (34) produce $Q(r) \propto r^{2/3-4}$ at sufficiently small r in the inertial range. If so, then $D_p(r)$ would asymptotically become $r^{2/3}$. The asymptotic inertial-range formula appears to be very sensitive to the exact cancellation of the derivatives in the summations (57) and (58).

The compressibility effects can be expressed in terms of the nonvanishing of the quantity $M_{ijkl}(\vec{r})_{|ijkl}$. Like $D_{ijkl}(\vec{r})$, $M_{ijkl}(\vec{r})$ is symmetric under interchange of any pair of its indices. Consequently, with the symbol D replaced by M , the results in Tables A1 and C1 all hold as do (A14) to (A16), and (A20) to (A22). Equation (18) is replaced by

$$D_{ijkl}(\vec{r})_{|ijkl} - M_{ijkl}(\vec{r})_{|ijkl} = 6Q(r). \quad (68)$$

The derivations of Appendix D and Sec. 5 hold when using $M_{ijkl}(\vec{r})$ in place of $D_{ijkl}(\vec{r})$. Consequently, in addition to (62), the pressure structure function has a new term given by

$$\begin{aligned}
M_p(r) = & \frac{1}{3} M_{1111}(r) - \frac{4}{3} r^2 \int_r^\infty y^{-3} [M_{1111}(y) + M_{\lambda\lambda\lambda}(y) - 6 M_{11\gamma\gamma}(y)] dy \\
& - \frac{4}{3} \int_0^r y^{-1} [M_{\lambda\lambda\lambda}(y) - 3 M_{11\gamma\gamma}(y)] dy.
\end{aligned} \tag{69}$$

Note that we obtain this without using the *SI* or *JG* assumptions. Of course, $M_p(r)$ is not the only additional term contributing to the pressure structure function; to determine all the others, we would have to begin with the equations for compressible fluid flow.

In Appendix J, we show that (69) requires that the incompressibility condition be more accurate for greater Reynolds numbers and also for smaller r/L_0 , where r is a spacing in the inertial range.

8. SUMMARY

In this paper, we derive the relationship between the pressure structure function and the fourth-order velocity structure function $D_{ijkl}(\vec{r})$; the pressure spectrum and correlation therefore are also related to $D_{ijkl}(\vec{r})$. We treat the simplest case of incompressibility and isotropic turbulence. The previous formulation related the pressure correlation and spectrum to the fourth-order velocity correlation $R_{ijkl}(\vec{r})$; that $R_{ijkl}(\vec{r})$ is not the most appropriate statistic to use is clear from several considerations. First, there must be subtraction of very large values of $R_{ijkl}(\vec{r})$ to produce the relatively small quantities needed to obtain pressure correlations, spectra, and structure functions. This implies that measurements of $R_{ijkl}(\vec{r})$ would have to be extremely precise to produce the pressure quantities. This problem becomes increasingly severe as length scale is decreased within the inertial and dissipation ranges for very large Reynolds numbers. Second, $R_{ijkl}(\vec{r})$ depends on the energy-containing range, whereas both $D_p(r)$ and $D_{ijkl}(\vec{r})$ should exhibit local isotropy. Thus, by relating $D_p(r)$ to $D_{ijkl}(\vec{r})$, we have a useful relationship for the locally isotropic case, even if the energy-containing range is anisotropic and inhomogeneous.

The *JG* approximation was previously used to obtain results for the pressure quantities. It is now clear [from (16), (B1), (57), and (58)] that the combination of incompressibility and the *JG* approximation were used to eliminate the very large, extraneous parts of $R_{ijkl}(\vec{r})$. In Sec. 5, we find that the *JG* approximation is implausibly accurate for prediction of the inertial range of $D_p(r)$. We derive $D_p(r)$ without use of the *JG* assumption or any replacement approximation. This makes our method the natural beginning point for studies of turbulence of compressible fluids and of anisotropic turbulence, such as atmospheric turbulence. We validate our derivations by showing that our formulas become their previously known *JG* versions when the *JG* assumption is applied.

One component of $D_{ijkl}(\vec{r})$ has been the subject of extensive experimental and theoretical investigation. As a result, for the inertial range we obtain $D_p(r) \propto r^{4/3}$, as derived by Obukhov (1949) and Batchelor (1951), or perhaps slightly different from $r^{4/3}$, as indicated in (59a,b). The inertial-range proportionality factor given in (C13) or (61) has three terms involving the level of three components of the structure function. This factor is sensitive to the cancellation between the three terms. As a result, this factor is sensitive to the accuracy of local isotropy in any given data, and this factor is probably sensitive to the inertial-range exponent of the structure-function components, and it is probably different from the proportionality factor derived by Obukhov and by Batchelor. The three structure function components should be measured or obtained from numerical simulation of the Navier-Stokes equation; that $D_p(r) > 0$ places bounds on the relative magnitudes of these components in the inertial and viscous ranges.

We present formulas for the mean-squared pressure gradient χ and pressure variance in Appendices G and H, respectively. Study of inner scales of second- and fourth-order velocity structure functions in Appendix F leads to estimates of χ and its JG version χ^{JG} . We find that χ/χ^{JG} depends on the Reynolds number, so χ^{JG} is of limited applicability.

Incompressibility is important in our derivations. In Sec. 6 and Appendix J, we estimate the constraint on compressibility such that the present results for $D_p(r)$ remain valid.

The acceleration correlation is important in studies of particle dispersion, aerosol coagulation, and sound radiated by bubbles in turbulence. In Appendix K, we give expressions for the acceleration correlation based on (62).

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10. REFERENCES

- Anselmet, F., Y. Gagne, E. J. Hopfinger, and R. A. Antonia, 1984. High-order velocity structure functions in turbulent shear flow. *J. Fluid Mech.* **140**:63-89.
- Antonia, R. A., B. R. Satyaprakash, and A. J. Chambers, 1982a. Reynolds number dependence of velocity structure functions in turbulent shear flows. *Phys. Fluids* **25**:29-37.

- Antonia, R. A., S. Rajagopalan, L. W. B. Browne, and A. J. Chambers, 1982b. Correlations of squared velocity and temperature derivatives in a turbulent plane jet. *Phys. Fluids* **25**:1156-1158.
- Batchelor, G. K., 1951. Pressure fluctuations in isotropic turbulence. *Proc., Camb. Philos. Soc.* **47**:359-374.
- Batchelor, G. K., 1960. *The Theory of Homogeneous Turbulence*. Cambridge University Press, Cambridge, England, 197 pp.
- George, W. K., P. D. Beuther, and R. E. A. Arndt, 1984. Pressure spectra in turbulent free shear flows. *J. Fluid Mech.* **148**:155-191.
- Hill, R. J., 1994. Corrections to Taylor's frozen turbulence approximation (unpublished manuscript), 48 pp.
- Karyakin, M. Yu., V. R. Kuznetsov, and A. A. Praskovskii, 1991. Experimental verification of the hypothesis of fine-scale isotropy of turbulence. *Fluid Dyn.* **26**:658-670.
- Kolmogorov, A. N., 1941. The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. *Dokl. Akad. Nauk. SSSR* **30**:301-305.
- Kolmogorov, A. N., 1962. A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number. *J. Fluid Mech.* **13**:82-85.
- Lumley, J. L., 1965. Interpretation of time spectra measured in high-intensity shear flows. *Phys. Fluids* **8**:1056-1062.
- Monin, A. S., and A. M. Yaglom, 1975. *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. 2. The MIT Press, Cambridge, Massachusetts, 874 pp.
- Obukhov, A. M., 1949. Pressure fluctuations in turbulent flow. *C. R. Akad. Sci. URSS* **66**:17-20.
- Obukhov, A. M., and A. M. Yaglom, 1951. The microstructure of turbulent flow. *Prikl. Mat. Mekh.* **15**:3-26. (Translation in NACA TM 1350, National Advisory Committee for Aeronautics, June 1953.)
- Sreenivasan, K. R., and P. Kailasnath, 1993. An update on the intermittency exponent in turbulence. *Phys. Fluids* **A5**:512-514.
- Tatarskii, V. I., 1971. *The Effects of the Turbulent Atmosphere on Wave Propagation*. Keter Press, Jerusalem, 471 pp.

- Uberoi, M. S., 1953. Quadruple velocity correlations and pressure fluctuations in isotropic turbulence. *J. Aero. Sci.* **20**:197-204.
- Van Atta, C. W., and W. Y. Chen, 1970. Structure functions of turbulence in the atmospheric boundary layer over the ocean. *J. Fluid Mech.* **44**:145-159.
- Van Atta, C. W., and J. Park, 1972. Statistical self-similarity and inertial subrange turbulence. In *Statistical Models and Turbulence*. M. Rosenblatt and C. Van Atta (Eds.), Lecture Notes in Physics, vol. 12, Springer Verlag, Berlin, 402-426.
- Van Atta, C. W., and J. C. Wyngaard, 1975. On higher-order spectra of turbulence. *J. Fluid Mech.* **72**:673-694.
- Vasilenko, V. M., M. M. Lyubimtsev, and R. V. Osmidov, 1975. Fluctuations of the turbulent energy dissipation rate and of the higher-order structure functions of the velocity field in the ocean. *Izv. Acad. Sci. USSR, Atmos. Oceanic Phys.* **11**:580-584.
- Wyngaard, J. C., and S. F. Clifford, 1977. Taylor's hypothesis and high-frequency turbulence spectra. *J. Atmos. Sci.* **34**:922-929.
- Wyngaard, J. C., and H. Tennekes, 1970. Measurements of the small-scale structure of turbulence at moderate Reynolds numbers. *Phys. Fluids* **13**:1962-1969.
- Yaglom, A. M., 1981. Laws of small-scale turbulence in atmosphere and ocean (In commemoration of the 40th anniversary of the theory of locally isotropic turbulence). *Izv. Acad. Sci. USSR, Atmos. Oceanic Phys.* **17**:919-935.

Appendix A: Isotropic Tensors

For isotropic turbulence, the general form of a fourth-order statistical tensor can be written as [following from Eq. (3.3.7) by Batchelor (1960)]

$$\begin{aligned}
 Z_{ijkl}(\vec{r}) \equiv & A_z(r) \frac{r_i r_j r_k r_l}{r^4} + B_z(r) \frac{r_i r_j}{r^2} \delta_{kl} + C_z(r) \frac{r_j r_k}{r^2} \delta_{il} \\
 & + D_z(r) \frac{r_k r_l}{r^2} \delta_{ij} + E_z(r) \frac{r_l r_i}{r^2} \delta_{jk} + F_z(r) \frac{r_i r_k}{r^2} \delta_{jl} \\
 & + G_z(r) \frac{r_j r_l}{r^2} \delta_{ik} + H_z(r) \delta_{ij} \delta_{kl} + I_z(r) \delta_{ik} \delta_{jl} \\
 & + J_z(r) \delta_{il} \delta_{jk}.
 \end{aligned} \tag{A1}$$

The $A_z(r)$, $B_z(r)$, ..., $J_z(r)$ are scalar functions of only $r = |\vec{r}|$. This tensor can be applied to correlations or structure functions.

We perform the fourth-order differentiation on (A1) to obtain

$$\begin{aligned}
 Z_{ijkl}(\vec{r})_{|ijkl} = & A_z^{(4)}(r) + 8 \frac{A_z^{(3)}(r)}{r} + 12 \frac{A_z^{(2)}(r)}{r^2} \\
 & + K_z^{(4)}(r) + 6 \frac{K_z^{(3)}(r)}{r} + 2 \frac{K_z^{(2)}(r)}{r^2} - 4 \frac{K_z^{(1)}(r)}{r^3} \\
 & + 4 \frac{K_z(r)}{r^4} + L_z^{(4)}(r) + 4 \frac{L_z^{(3)}(r)}{r},
 \end{aligned} \tag{A2}$$

where the superscript numeral in parentheses indicates the order of differentiation with respect to r , and

$$K_z(r) \equiv B_z(r) + C_z(r) + D_z(r) + E_z(r) + F_z(r) + G_z(r) \tag{A3}$$

$$L_z(r) \equiv H_z(r) + I_z(r) + J_z(r). \tag{A4}$$

An important labor-saving identity for obtaining (A2) from (A1) is

$$r_i(r_j/r)_{|i} = 0.$$

We define the operators

$$\nabla_m \equiv \frac{\partial}{\partial r} + \frac{m}{r},$$

where m is a real number. Then ∇_1^2 is the radial part of the Laplacian operator in spherical coordinates. Operators ∇_m and ∇_n commute only if $m = n$. Now, (A2) can be written more succinctly as

$$Z_{ijkl}(\vec{r})_{ijkl} = \nabla_2^4 A_z(r) + \nabla_1^2 \nabla_2^2 K_z(r) + \nabla_1^4 L_z(r).$$

Applying the commutative property of multiplication to (8) gives

$$\Delta_{ijkl} = \Delta_{jikl} \tag{A5}$$

$$\Delta_{ijkl} = \Delta_{ijlk} \tag{A6}$$

$$\Delta_{ijkl} = \Delta_{klij}. \tag{A7}$$

When applied to (A1), (A5) to (A7) give

$$B_\Delta = D_\Delta \tag{A8}$$

$$C_\Delta = E_\Delta = F_\Delta = G_\Delta \tag{A9}$$

$$I_\Delta = J_\Delta. \tag{A10}$$

Therefore, (A2) applies with

$$K_\Delta = 2B_\Delta + 4C_\Delta \tag{A11}$$

$$L_\Delta = H_\Delta + 2I_\Delta. \tag{A12}$$

Our chosen coordinate system is Cartesian with the 1-axis along the separation vector \vec{r} . In this case, $r_1 = r$ and $r_i/r = \delta_{1i}$, so (A1) and (A8) to (A10) give

$$\begin{aligned}
\Delta_{ijkl}(\vec{r}) = & A_{\Delta}(r) \delta_{1i} \delta_{1j} \delta_{1k} \delta_{1l} + B_{\Delta}(r) [\delta_{1i} \delta_{1j} \delta_{kl} + \delta_{1k} \delta_{1l} \delta_{ij}] \\
& + C_{\Delta}(r) [\delta_{1j} \delta_{1k} \delta_{il} + \delta_{1i} \delta_{1l} \delta_{jk} + \delta_{1i} \delta_{1k} \delta_{jl} + \delta_{1j} \delta_{1l} \delta_{ik}] \quad (A13) \\
& + H_{\Delta}(r) \delta_{ij} \delta_{kl} + I_{\Delta}(r) [\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}].
\end{aligned}$$

This tensor has only five linearly independent components. Table A1 gives the coefficients of the functions, A_{Δ} , B_{Δ} , C_{Δ} , H_{Δ} , I_{Δ} for all nonvanishing components of Δ_{ijkl} .

Table A1.--Coefficients of $A_{\Delta}(r)$, $B_{\Delta}(r)$, $C_{\Delta}(r)$, $H_{\Delta}(r)$, and $I_{\Delta}(r)$ in (A13) and of $A_D(r)$, $B_D(r)$, and $H_D(r)$ in (A14). Coefficients that are all zero are not listed.

i	j	k	l	A_{Δ}	B_{Δ}	C_{Δ}	H_{Δ}	I_{Δ}	A_D	B_D	H_D
1	1	1	1	1	2	4	1	2	1	6	3
1	1	2	2	0	1	0	1	0	0	1	1
1	1	3	3	0	1	0	1	0	0	1	1
1	2	1	2	0	0	1	0	1	0	1	1
1	2	2	1	0	0	1	0	1	0	1	1
1	3	1	3	0	0	1	0	1	0	1	1
1	3	3	1	0	0	1	0	1	0	1	1
2	1	1	2	0	0	1	0	1	0	1	1
2	1	2	1	0	0	1	0	1	0	1	1
2	2	1	1	0	1	0	1	0	0	1	1
2	2	2	2	0	0	0	1	2	0	0	3
2	2	3	3	0	0	0	1	0	0	0	1
2	3	2	3	0	0	0	0	1	0	0	1
2	3	3	2	0	0	0	0	1	0	0	1
3	1	1	3	0	0	1	0	1	0	1	1
3	1	3	1	0	0	1	0	1	0	1	1
3	2	2	3	0	0	0	0	1	0	0	1
3	2	3	2	0	0	0	0	1	0	0	1
3	3	1	1	0	1	0	1	0	0	1	1
3	3	2	2	0	0	0	1	0	0	0	1
3	3	3	3	0	0	0	1	2	0	0	3

Since D_{ijkl} , defined in (9), is symmetric under interchange of all its indices, its isotropic form corresponding to (A1) has $B_D = C_D = D_D = E_D = F_D = G_D$ and $H_D = I_D = J_D$.

The formula analogous to (A13) is

$$\begin{aligned}
D_{ijkl}(\vec{r}) = & A_D(r) \delta_{1i} \delta_{1j} \delta_{1k} \delta_{1l} \\
& + B_D(r) [\delta_{1i} \delta_{1j} \delta_{kl} + \delta_{1k} \delta_{1l} \delta_{ij} + \delta_{1j} \delta_{1k} \delta_{il} \\
& \quad + \delta_{1i} \delta_{1l} \delta_{jk} + \delta_{1i} \delta_{1k} \delta_{jl} + \delta_{1j} \delta_{1l} \delta_{ik}] \\
& + H_D(r) [\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}].
\end{aligned} \tag{A14}$$

The coefficients of A_D , B_D , and H_D are given in Table A1 for all of the nonvanishing D_{ijkl} . From (A11) and (A12), we see that (A2) applies to $D_{ijkl}(\vec{r})|_{ijkl}$ with

$$K_D(r) = 6 B_D(r) \tag{A15}$$

and

$$L_D(r) = 3 H_D(r). \tag{A16}$$

From (A13), the nonvanishing components are

$$\Delta_{\alpha\alpha\alpha\alpha}(r) = [A_\Delta(r) + 2 B_\Delta(r) + 4 C_\Delta(r)] \delta_{1\alpha} + H_\Delta(r) + 2 I_\Delta(r) \tag{A17}$$

$$\Delta_{\alpha\alpha\beta\beta}(r) = B_\Delta(r) (\delta_{1\alpha} + \delta_{1\beta}) + H_\Delta(r) \quad \text{for } \alpha \neq \beta \tag{A18}$$

$$\Delta_{\alpha\beta\alpha\beta}(r) = C_\Delta(r) (\delta_{1\alpha} + \delta_{1\beta}) + I_\Delta(r) \quad \text{for } \alpha \neq \beta. \tag{A19}$$

From (A14), the nonvanishing components are

$$D_{\alpha\alpha\alpha\alpha}(r) = [A_D(r) + 6 B_D(r)] \delta_{1\alpha} + 3 H_D(r) \tag{A20}$$

$$D_{\alpha\alpha\beta\beta}(r) = B_D(r) (\delta_{1\alpha} + \delta_{1\beta}) + H_D(r) \quad \text{for } \alpha \neq \beta. \tag{A21}$$

Equations (A17) to (A21) determine the functions A_Δ , B_Δ , C_Δ , H_Δ , I_Δ and A_D , B_D , H_D in terms of measurable, linearly independent components of the fourth-order structure functions. Substituting (A15), (A16), (A20), and (A21) into (A2) gives

$$\begin{aligned}
D_{ijkl}(\vec{r})_{|ijkl} &= D_{1111}^{(4)}(r) + \frac{8}{r} D_{1111}^{(3)}(r) + \frac{12}{r^2} D_{1111}^{(2)}(r) \\
&\quad - \frac{12}{r} D_{11\gamma\gamma}^{(3)}(r) - \frac{60}{r^2} D_{11\gamma\gamma}^{(2)}(r) - \frac{24}{r^3} D_{11\gamma\gamma}^{(1)}(r) \\
&\quad + \frac{24}{r^4} D_{11\gamma\gamma}(r) + \frac{8}{r^2} D_{\lambda\lambda\lambda\lambda}^{(2)}(r) \\
&\quad + \frac{8}{r^3} D_{\lambda\lambda\lambda\lambda}^{(1)}(r) - \frac{8}{r^4} D_{\lambda\lambda\lambda\lambda}(r).
\end{aligned} \tag{A22}$$

In (A22), γ is 2 or 3, and λ is 2 or 3; γ can be taken to be equal to λ , or different from λ . Define κ to be 2 or 3, but $\kappa \neq \lambda$; then from (A20) and (A21), we have $3 D_{\lambda\lambda\kappa\kappa}(r) = D_{\lambda\lambda\lambda\lambda}(r)$; one could therefore substitute $3 D_{\lambda\lambda\kappa\kappa}(r)$ anywhere $D_{\lambda\lambda\lambda\lambda}(r)$ appears in (A22). Equation (A22) can be written more succinctly as

$$D_{ijkl}(\vec{r})_{|ijkl} = \nabla_2^4 D_{1111}(r) - \frac{12}{r} \nabla_1 \nabla_2^2 D_{11\gamma\gamma}(r) + \frac{8}{r^2} \nabla_0 \nabla_1 D_{\lambda\lambda\lambda\lambda}(r).$$

The general form of a second-order isotropic tensor is (Batchelor, 1960)

$$Z_{ij}(\vec{r}) = S_z(r) \frac{r_i r_j}{r^2} + T_z(r) \delta_{ij}. \tag{A23}$$

Applied to $D_{ij}(\vec{r})$ for the chosen coordinate system, we can write

$$D_{ij}(\vec{r}) = [D_{11}(r) - D_{22}(r)] \delta_{1i} \delta_{1j} + D_{22}(r) \delta_{ij}. \tag{A24}$$

The only nonvanishing components are $D_{11}(r)$, $D_{22}(r)$, and $D_{33}(r)$ with $D_{22}(r) = D_{33}(r)$. The condition of incompressibility, (57) and (58), gives the relationship (Obukhov, 1949)

$$\frac{2}{r} D_{22}(r) = D_{11}^{(1)}(r) + \frac{2}{r} D_{11}(r) = \nabla_2 D_{11}(r). \tag{A25}$$

Analogous relationships hold for $R_{ij}(\vec{r})$, as given by Batchelor (1960).

Appendix B: Moments of the Joint Gaussian Distribution

Here, we give the relationship of fourth-order correlations (and structure functions) to second-order correlations (and structure functions) that are obtained from the joint Gaussian assumption. The relationship for $R_{ijkl}^{JG}(\vec{r})$, defined in (7), is given by Batchelor (1960) as

$$R_{ijkl}^{JG}(\vec{r}) = \sigma_{ij}\sigma_{kl} + R_{ik}(\vec{r})R_{jl}(\vec{r}) + R_{il}(\vec{r})R_{jk}(\vec{r}). \quad (\text{B1})$$

Substituting (21) in (B1) gives

$$\begin{aligned} R_{ijkl}^{JG}(\vec{r}) &= \sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk} \\ &\quad - \frac{1}{2} [\sigma_{ik}D_{jl}(\vec{r}) + \sigma_{jl}D_{ik}(\vec{r}) + \sigma_{il}D_{jk}(\vec{r}) + \sigma_{jk}D_{il}(\vec{r})] \\ &\quad + \frac{1}{4} [D_{ik}(\vec{r})D_{jl}(\vec{r}) + D_{il}(\vec{r})D_{jk}(\vec{r})]. \end{aligned} \quad (\text{B2})$$

The moment $\langle u'_i u_j u_k u_l \rangle^{JG}$ is easily obtained from the moment-generating function of the joint Gaussian distribution, which is in Sec. 8.3 by Batchelor (1960). We obtain

$$\langle u'_i u_j u_k u_l \rangle^{JG} = \sigma_{kl}R_{ij}(\vec{r}) + \sigma_{jl}R_{ik}(\vec{r}) + \sigma_{jk}R_{il}(\vec{r}). \quad (\text{B3})$$

Substituting into (13) the moments obtained by cyclic permutations of the indices in (B3) gives

$$\begin{aligned} M_{ijkl}^{JG}(\vec{r}) &= 4 \left[2(\sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}) - \sigma_{ij}R_{kl}(\vec{r}) - \sigma_{kl}R_{ij}(\vec{r}) \right. \\ &\quad \left. - \sigma_{jl}R_{ik}(\vec{r}) - \sigma_{ik}R_{jl}(\vec{r}) - \sigma_{jk}R_{il}(\vec{r}) - \sigma_{il}R_{jk}(\vec{r}) \right] \end{aligned} \quad (\text{B4})$$

$$\begin{aligned} &= 2 \left[\sigma_{ij}D_{kl}(\vec{r}) + \sigma_{kl}D_{ij}(\vec{r}) + \sigma_{jl}D_{ik}(\vec{r}) \right. \\ &\quad \left. + \sigma_{ik}D_{jl}(\vec{r}) + \sigma_{jk}D_{il}(\vec{r}) + \sigma_{il}D_{jk}(\vec{r}) \right], \end{aligned} \quad (\text{B5})$$

where (B5) results from substituting (21) in (B4).

The formula for $\Delta_{ijkl}^{JG}(\vec{r})$ follows trivially from (10) and (B1) or (B2):

$$\Delta_{ijkl}^{JG}(\vec{r}) = 2 \left[\sigma_{ik} \sigma_{jl} + \sigma_{il} \sigma_{jk} - R_{ik}(\vec{r}) R_{jl}(\vec{r}) - R_{il}(\vec{r}) R_{jk}(\vec{r}) \right] \quad (\text{B6})$$

$$\begin{aligned} &= \sigma_{il} D_{jk}(\vec{r}) + \sigma_{jk} D_{il}(\vec{r}) + \sigma_{ik} D_{jl}(\vec{r}) + \sigma_{jl} D_{ik}(\vec{r}) \\ &\quad - \frac{1}{2} \left[D_{il}(\vec{r}) D_{jk}(\vec{r}) + D_{ik}(\vec{r}) D_{jl}(\vec{r}) \right]. \end{aligned} \quad (\text{B7})$$

Now, $D_{ijkl}^{JG}(\vec{r})$ is easily obtained from (11) or (12). We substitute (B2) and (B5) in (11); all terms like $\sigma_{ij} D_{kl}(\vec{r})$ that appear in (B2) and (B5) cancel:

$$D_{ijkl}^{JG}(\vec{r}) = D_{ik}(\vec{r}) D_{jl}(\vec{r}) + D_{il}(\vec{r}) D_{jk}(\vec{r}) + D_{ij}(\vec{r}) D_{kl}(\vec{r}). \quad (\text{B8})$$

Only the asymptotically smallest terms contribute in (B8).

From (B7) we obtain the nonvanishing components (see Table A1) in the chosen coordinate system:

$$\Delta_{\alpha\alpha\beta\beta}^{JG}(r) = 4 \sigma_{\alpha\beta} D_{\alpha\beta}(r) - [D_{\alpha\beta}(r)]^2 \quad (\text{B9})$$

$$\begin{aligned} \Delta_{\alpha\beta\alpha\beta}^{JG}(r) &= \sigma_{\alpha\alpha} D_{\beta\beta}(r) + 2 \sigma_{\alpha\beta} D_{\alpha\beta}(r) + \sigma_{\beta\beta} D_{\alpha\alpha}(r) \\ &\quad - \frac{1}{2} \left\{ D_{\alpha\alpha}(r) D_{\beta\beta}(r) + [D_{\alpha\beta}(r)]^2 \right\}. \end{aligned} \quad (\text{B10})$$

Both (B9) and (B10) give

$$\Delta_{\alpha\alpha\alpha\alpha}^{JG}(r) = 4 \sigma_{\alpha\alpha} D_{\alpha\alpha}(r) - [D_{\alpha\alpha}(r)]^2. \quad (\text{B11})$$

Since (B11) renders (B9) and (B10) irrelevant unless $\alpha \neq \beta$, we state that

$$\Delta_{\alpha\alpha\beta\beta}^{JG}(r) = 0 \quad \text{for } \alpha \neq \beta \quad (\text{B12})$$

$$\begin{aligned} \Delta_{\alpha\beta\alpha\beta}^{JG}(r) &= \sigma_{\alpha\alpha} D_{\beta\beta}(r) + \sigma_{\beta\beta} D_{\alpha\alpha}(r) \\ &\quad - \frac{1}{2} D_{\alpha\alpha}(r) D_{\beta\beta}(r) \quad \text{for } \alpha \neq \beta. \end{aligned} \quad (\text{B13})$$

Comparing (B11) to (B13) with (A17) to (A19) relates the functions A_{Δ}^{JG} , B_{Δ}^{JG} , C_{Δ}^{JG} , H_{Δ}^{JG} , I_{Δ}^{JG} to variances and second-order structure functions. In particular, (B12) and (A18) require

$$B_{\Delta}^{JG}(r) = 0 \quad \text{and} \quad H_{\Delta}^{JG}(r) = 0. \quad (\text{B14})$$

Similarly, from (B8) we obtain the nonvanishing components

$$D_{\alpha\alpha\alpha\alpha}^{JG}(r) = 3 [D_{\alpha\alpha}(r)]^2 \quad (\text{B15})$$

$$D_{\alpha\alpha\beta\beta}^{JG}(r) = D_{\alpha\alpha}(r) D_{\beta\beta}(r) \quad \text{for } \alpha \neq \beta. \quad (\text{B16})$$

Comparing (B15) and (B16) with (A20) and (A21) relates the functions A_D^{JG} , B_D^{JG} , and H_D^{JG} to second-order structure functions. Let $\lambda = 2$ or 3 ; then we have

$$A_{\Delta}^{JG}(r) = - [D_{11}(r) - D_{\lambda\lambda}(r)]^2 \quad (\text{B17})$$

$$K_{\Delta}^{JG}(r) = 4 C_{\Delta}^{JG}(r) = 2 [2\sigma_{\lambda\lambda} - D_{\lambda\lambda}(r)] [D_{11}(r) - D_{\lambda\lambda}(r)] \quad (\text{B18})$$

$$L_{\Delta}^{JG}(r) = 2 I_{\Delta}^{JG}(r) = 4 \sigma_{\lambda\lambda} D_{\lambda\lambda}(r) - [D_{\lambda\lambda}(r)]^2 \quad (\text{B19})$$

$$A_D^{JG}(r) = 3 [D_{11}(r) - D_{\lambda\lambda}(r)]^2 \quad (\text{B20})$$

$$K_D^{JG(r)} = 6 B_D^{JG}(r) = 6 D_{\lambda\lambda}(r) [D_{11}(r) - D_{\lambda\lambda}(r)] \quad (\text{B21})$$

$$L_D^{JG(r)} = 3 H_D^{JG}(r) = 3 [D_{\lambda\lambda}(r)]^2. \quad (\text{B22})$$

By incompressibility, the terms in (B17) to (B19) that are proportional to $\sigma_{\lambda\lambda}$ must vanish when substituted into (A2). We write these terms as

$$K \equiv 4 \sigma_{\lambda\lambda} [D_{11}(r) - D_{\lambda\lambda}(r)]$$

$$L \equiv 4 \sigma_{\lambda\lambda} D_{\lambda\lambda}(r).$$

Using (A25), it is easy to show that

$$\nabla_2^2 K + \nabla_1^2 L = 0.$$

Hence, these terms vanish from (A2) as they must. The remaining terms in (B17) to (B19) are related to (B20) to (B22) by a factor of -3, which is the factor relating (17) and (18). Hence, (A2) with (B17) to (B19) give the same $Q(r)$ as (A2) with (B20) to (B22), and hence generate the same $D_p^{JG}(r)$ as they must. It is simpler to first recast (A2) in a form analogous to (A22); then, using (B17) to (B19) or (B20) to (B22) after eliminating $D_{\lambda\lambda}(r)$ in favor of $D_{11}(r)$ by using the incompressibility condition (A25), we obtain

$$\begin{aligned}
 Q^{JG}(r) = & 2 \left[D_{11}^{(2)}(r) \right]^2 + 2 D_{11}^{(1)}(r) D_{11}^{(3)}(r) \\
 & + \frac{10}{r} D_{11}^{(1)}(r) D_{11}^{(2)}(r) + \frac{3}{r^2} \left[D_{11}^{(1)}(r) \right]^2
 \end{aligned} \tag{B23}$$

This is the same as Batchelor's (1951) Eq. (5.3), thereby validating our derivations.

Appendix C: The Inertial Range of $Q(r)$, and of the Pressure Structure Function and Its Spectrum

Whether we substitute (A2) or (A22) into (18), we have the formula

$$Q(r) = \frac{1}{6} D_{ijkl|ijkl}(r) = \frac{1}{6} \sum_{i=1}^3 \sum_{n=0}^4 W_{in} f_i^{(n)}(r) r^{n-4}, \quad (C1)$$

where W_{in} are the coefficients in (A2) or (A22), which are also given in Table C1, and $f_i(r)$ is a representation of the three structure functions in (A2) or (A22); $f_i^{(n)}(r)$ is the n th derivative of $f_i(r)$ with respect to r . There are many possible choices for the $f_i(r)$; three choices are given in the left column of Table C1. We hypothesize that in the inertial range we have

$$f_i(r) = C_i \varepsilon^{4/3} r^q. \quad (C2)$$

The notation C_i for the inertial-range coefficients in (C2) is for simplicity. The more descriptive notation C_{11} , $C_{1\gamma}$, and $C_{\lambda\lambda}$ for the inertial-range coefficients of $D_{1111}(r)$, $D_{11\gamma\gamma}(r)$, and $D_{\lambda\lambda\lambda\lambda}(r)$ is used elsewhere in this report. According to intermittency theory, $q = 4/3 - 2\mu/9$, and C_1 , C_2 , and C_3 depend on the turbulence macrostructure and have dimensions of a macroscale raised to the power $2\mu/9$. In the original similarity theory, $q = 4/3$ and C_1 , C_2 , and C_3 are universal constants. Any prediction for q could be used in (C2). We define

$$H_1 \equiv 1 \quad (C3)$$

$$H_2 \equiv C_2/C_1 = f_2(r)/f_1(r) \quad (C4)$$

$$H_3 \equiv C_3/C_1 = f_3(r)/f_1(r). \quad (C5)$$

Whether C_1 , C_2 , and C_3 are macrostructure-dependent or not, we hypothesize that H_2 and H_3 are universal constants. We expect that H_2 and H_3 are more accurately measurable than C_2 and C_3 . By repeated differentiation of (C2), we have

$$f_i^{(n)}(r) = C_i \varepsilon^{4/3} \Pi(q, n) r^{q-n}, \quad (C6)$$

where

$$\begin{aligned} \Pi(q, n) &\equiv (q-0)(q-1)(q-2)\dots(q-(n-1)) \quad \text{for } n = 1, 2, 3, \dots \\ \Pi(q, 0) &\equiv 1. \end{aligned} \quad (C7)$$

Using (C3) to (C7) in (C1) for the inertial range, we may thus write

$$Q(r) = C_Q \varepsilon^{4/3} r^{q-4}, \quad (C8)$$

where

$$C_Q \equiv C_1 \sum_{t=1}^3 H_t X_t(q), \quad (C9)$$

and wherein the numerical coefficients are

$$X_t(q) \equiv \frac{1}{6} \sum_{n=0}^4 W_{tn} \Pi(q, n), \quad (C10)$$

which are given in Table C1. Given measurements of C_1 , H_2 , and H_3 , the coefficient C_Q becomes known.

Table C1.--Coefficients W_{tn} in (A2) and (A22) and coefficients Y_{tn} in (D11) to (D15), for $n = 0$ to 4 and $t = 1$ to 3, for three choices of function sets f_t in (C1) (separated here by single horizontal lines). Also, $\Pi(q, n)$ and $X_t(q)$, defined in (C7) and (C10), for $q = 4/3$ and $n = 0$ to 4.

f_t	$t \backslash n$	0		1		2		3		4		$X_t(4/3)$
		W_{t0}	Y_{t0}	W_{t1}	Y_{t1}	W_{t2}	Y_{t2}	W_{t3}	Y_{t3}	W_{t4}	Y_{t4}	
D_{1111}	1	0	-1	0	0	12	0	8	0	1	4	0.576
$D_{11\gamma\gamma}$	2	24	0	-24	0	-60	-12	-12	0	0	-24	-5.19
$D_{\lambda\lambda\lambda\lambda}$	3	-8	0	8	0	8	4	0	0	0	4	1.04
D_{1111}	1	0	-1	0	0	12	0	8	0	1	4	0.576
$D_{11\gamma\gamma}$	2	0	24	0	0	-36	0	-12	0	0	-12	-2.07
$D_{11\gamma\gamma} - D_{\lambda\lambda\lambda\lambda}/3$	3	24	0	-24	0	-24	-12	0	0	0	-12	-3.11
A_D	1	0	-1	0	0	12	0	8	0	1	4	0.576
K_D	2	4	-1	-4	0	2	-2	6	0	1	0	-0.288
L_D	3	0	-1	0	0	0	0	4	0	1	0	-0.115
$\Pi\left(\frac{4}{3}, n\right)$		1		$\frac{4}{3}$		$\frac{4}{9}$		$-\frac{8}{27}$		$\frac{40}{81}$		

Substitution of (C8) into (3) gives our desired result,

$$D_p(r) = H_p C_{11} \varepsilon^{4/3} r^{-q} \quad (\text{C11})$$

$$= H_p D_{1111}(r), \quad (\text{C12})$$

where

$$H_p \equiv \frac{2}{q(2-q)(q^2-1)} \sum_{i=1}^3 H_i X_i(q) \quad (\text{C13a})$$

$$= \left[\frac{q}{3}(2+q) + \frac{8}{3} H_{\lambda\lambda} - 4(2+q) H_{1\gamma} \right] / [q(2-q)] \quad (\text{C13b})$$

$$\approx \frac{5}{3} \left(1 - \frac{2\mu}{5} \right) + 3 \left(1 - \frac{\mu}{6} \right) H_{\lambda\lambda} - 15 \left(1 - \frac{7\mu}{30} \right) H_{1\gamma}. \quad (\text{C13c})$$

In (C13a), the H_i can be ratios of any linear combination of the nonzero components of $D_{ijkl}(\vec{r})$, but in (C13b,c) we have chosen specific ratios defined by

$$H_{\lambda\lambda} \equiv D_{\lambda\lambda\lambda\lambda}(r) / D_{1111}(r),$$

$$H_{1\gamma} \equiv D_{11\gamma\gamma}(r) / D_{1111}(r),$$

where r is in the inertial range. The approximation in (C13c) is that only the lowest order in μ is retained.

The relationship between the structure function and the spatial spectrum from data along a line is (Tatarskii, 1971)

$$\Psi_p(k_1) = \frac{1}{\pi k_1^2} \int_0^{\infty} dr \cos(k_1 r) D_p^{(2)}(r). \quad (\text{C14})$$

We have chosen the relationship given by Tatarskii (1971) that converges for $1 < q < 2$. Our (C14) is twice Tatarskii's because we take the pressure variance, σ_p^2 , to be given by

$$\sigma_p^2 = \int_0^{\infty} \Psi_p(k_1) dk_1. \quad (\text{C15})$$

Substituting (C11) in (C14) gives, for $1 < q < 2$,

$$\Psi_p(k_1) = \pi^{-1} \Gamma(q+1) \cos\left[\frac{\pi}{2}(q-1)\right] H_p C_{11} \varepsilon^{4/3} k_1^{-q-1}. \quad (\text{C16})$$

For $q = 4/3$ and $C_p \equiv H_p C_{11}$, we have

$$\Psi_p(k_1) = 0.328 C_p \varepsilon^{4/3} k_1^{-7/3}. \quad (\text{C17})$$

Of course, the pressure spectrum corresponding to (E4) follows from (C17) by replacing C_p with $C^2 \approx 4$.

Appendix D: Reduction Using Integration by Parts

We now derive $D_p(r)$ from (3) and (A22). We use the representation of (A22) given in (C1); substituting this in (3) gives

$$D_p(r) = \frac{1}{18} \sum_{t=1}^3 \sum_{n=0}^4 W_{tn} F_{tn}(r), \quad (\text{D1})$$

where

$$\begin{aligned} F_{tn}(r) &= \frac{1}{r} \int_0^r dy (y^n - 3ry^{n-1} + 3r^2y^{n-2}) f_t^{(n)}(y) \\ &+ r^2 \int_r^\infty dy y^{n-3} f_t^{(n)}(y). \end{aligned} \quad (\text{D2})$$

We change integration variables to $z = y/r$. Differentiation of f_t is now with respect to z . We suppress the subscript t and the argument r , and $f^{(n)}$ is written where $f^{(n)}(z)$ appears. We delete writing dz after the integral symbol. Then, (D2) becomes

$$F_n = \int_0^1 (z^n - 3z^{n-1} + 3z^{n-2}) f^{(n)} + \int_1^\infty z^{n-3} f^{(n)}. \quad (\text{D3})$$

Repeated integration by parts gives

$$\begin{aligned} \int z^i f^{(n)} &= z^i f^{(n-1)} - i \int z^{i-1} f^{(n-1)} \\ &= \dots - iz^{i-1} f^{(n-2)} + i(i-1) \int z^{i-2} f^{(n-2)} \\ &= \dots + i(i-1) z^{i-2} f^{(n-3)} - i(i-1)(i-2) \int z^{i-3} f^{(n-3)} \\ &= \dots - i(i-1)(i-2) z^{i-3} f^{(n-4)} + i(i-1)(i-2)(i-3) \int z^{i-4} f^{(n-4)}, \end{aligned} \quad (\text{D4})$$

where ... means include all of the integrated terms that appear on all the lines above. These formulas can be applied to all of the integrals in (D3) that sum to give F_n . We do so, taking care that the remaining integrals are convergent and that the evaluated integrated terms are finite. We obtain

$$F_0 = \int_0^1 (1 - 3z^{-1} + 3z^{-2})f + \int_1^\infty z^{-3}f \quad (D5)$$

$$F_1 = \int_0^1 (-1 + 3z^{-2})f + 2 \int_1^\infty z^{-3}f \quad (D6)$$

$$F_2 = 2 \int_0^1 f + 2 \int_1^\infty z^{-3}f \quad (D7)$$

$$F_3 = -6 \int_0^1 f \quad (D8)$$

$$F_4 = -6f(1) + 24 \int_0^1 f. \quad (D9)$$

Substituting these into (D1), we obtain

$$D_p(r) = \frac{1}{3} \sum_{t=1}^3 \left(Y_{t0} f(1) + Y_{t1} \int_0^1 f + Y_{t2} \int_0^1 z^{-1}f \right. \\ \left. + Y_{t3} \int_0^1 z^{-2}f + Y_{t4} \int_1^\infty z^{-3}f \right), \quad (D10)$$

where

$$Y_{t0} = -W_{t4} \quad (D11)$$

$$Y_{t1} = (W_{t0} - W_{t1} + 2W_{t2} - 6W_{t3} + 24W_{t4})/6 \quad (D12)$$

$$Y_{t2} = -W_{t0}/2 \quad (D13)$$

$$Y_{t3} = (W_{t0} + W_{t1})/2 \quad (D14)$$

$$Y_{t4} = (W_{t0} + 2W_{t1} + 2W_{t2})/6. \quad (D15)$$

These coefficients are given in Table C1.

We can check (D10) by verifying that it gives the same inertial-range formula as (C11) and (C12). By substituting (C2) in (D10) and performing the integrals, we obtain this verification if

$$\left[q(2-q)(q^2-1) \right]^{-1} \sum_{n=0}^4 W_{tn} \Pi(q,n) = Y_{t0} + Y_{t1}(q+1)^{-1} + Y_{t2}q^{-1} + Y_{t3}(q-1)^{-1} + Y_{t4}(2-q)^{-1}. \quad (\text{D16})$$

Substituting (D11) to (D15) and (C7) in (D16), we have verified that each coefficient of W_{tn} for $n = 0$ to 4 are the same on each side of (D16).

Changing the integration variable back to y and displaying the full dependence on all parameters and using the fact that Table C1 shows that $Y_{t1} = Y_{t3} = 0$, (D10) becomes

$$D_p(r) = \frac{1}{3} \sum_{t=1}^3 \left[Y_{t0} f_t(r) + Y_{t2} \int_0^r y^{-1} f_t(y) dy + Y_{t4} r^2 \int_r^\infty y^{-3} f_t(y) dy \right]. \quad (\text{D17})$$

Appendix E: Pressure Structure Function from the JG Assumption

We present a (corrected) general formula for $D_p^{JG}(r)$ and its inertial-range formula including intermittency effects. We verify that our general formula (62) becomes Batchelor's (1951) formula when the JG approximation is used. Batchelor (1951) shows that the JG assumption simplifies (3) to give

$$D_p^{JG}(r) = \int_0^r y \left[D_{11}^{(1)}(y) \right]^2 dy + r^2 \int_r^\infty y^{-1} \left[D_{11}^{(1)}(y) \right]^2 dy. \quad (\text{E1})$$

Here, we expressed Batchelor's (1951) Eq. (6.4) in terms of the velocity structure function rather than the velocity correlation, and we corrected a sign error. The incompressibility condition (A25) allows the integrands in (E1) to be written in many different ways.

Consider the power-law inertial range such that

$$D_{11}(r) = C \varepsilon^{2/3} r^g, \quad (\text{E2})$$

where $0 < g < 1$. The original similarity hypothesis by Kolmogorov (1941) predicts $g = 2/3$ and that C is a universal constant, empirically $C \approx 2$. The intermittency theory by Kolmogorov (1962) predicts $g = 2/3 + \mu/9$ and that C may depend on turbulence macrostructure. Substituting (E2) in (E1), the inertial range of the pressure structure function is given by

$$D_p^{JG}(r) = \frac{g}{2(1-g)} C^2 \varepsilon^{4/3} r^{2g} = \frac{g}{2(1-g)} \left[D_{11}(r) \right]^2. \quad (\text{E3})$$

For $g = 2/3$, this is

$$D_p^{JG}(r) = C^2 \varepsilon^{4/3} r^{4/3} = \left[D_{11}(r) \right]^2, \quad (\text{E4})$$

as given previously by Obukhov (1949) and Batchelor (1951). For $g = 2/3 + \mu/9$, it is

$$D_p^{JG}(r) = \frac{1 + \frac{\mu}{6}}{1 - \frac{\mu}{3}} C^2 \varepsilon^{4/3} r^{4/3 + 2\mu/9} = \frac{1 + \frac{\mu}{6}}{1 - \frac{\mu}{3}} \left[D_{11}(r) \right]^2. \quad (\text{E5})$$

We note that the ratio of (E5) to (60) is approximately H_p times the flatness factor. The difference of the exponents $(4/3) + (2\mu/9)$ in (E5) and $(4/3) - (2\mu/9)$ in (59) is $4\mu/9 \approx 0.1$. Thus, compared with our theory, the theory based on the *JG* assumption by Obukhov (1949) and Batchelor (1951) produces a similar, but slightly different, inertial-range power law, and a perhaps greatly different coefficient.

To show that our general formula (62) reduces to (E1) under the *JG* assumption, we first rewrite the integrands in (E1) using the following formulas:

$$\left[D_{11}^{(1)}(r) \right]^2 = -\frac{1}{r} \left\{ [D_{11}(r)]^2 \right\}^{(1)} + \frac{4}{r^2} \left\{ [D_{\lambda\lambda}(r)]^2 - D_{11}(r) D_{\gamma\gamma}(r) \right\} \quad (\text{E6})$$

$$\left[D_{11}^{(1)}(r) \right]^2 = \frac{4}{r^2} \left\{ [D_{11}(r)]^2 + [D_{\lambda\lambda}(r)]^2 - 2 D_{11}(r) D_{\gamma\gamma}(r) \right\}. \quad (\text{E7})$$

Equations (E6) and (E7) are obtained using incompressibility (A25). As before, λ and γ are 2 or 3; we emphasize that the index γ need not be the same as the index λ . We substitute (E6) and (E7) in the first and second integrals in (E1), respectively; we obtain

$$\begin{aligned} D_P^{JG}(r) = & - [D_{11}(r)]^2 + 4 r^2 \int_r^\infty y^{-3} \left\{ [D_{11}(y)]^2 + [D_{\lambda\lambda}(y)]^2 - 2 D_{11}(y) D_{\gamma\gamma}(y) \right\} dy \\ & + 4 \int_0^r y^{-1} \left\{ [D_{\lambda\lambda}(y)]^2 - D_{11}(y) D_{\gamma\gamma}(y) \right\} dy. \end{aligned} \quad (\text{E8})$$

By substituting the *JG* relationships (B15) and (B16), (E8) becomes

$$\begin{aligned} D_P^{JG}(r) = & -\frac{1}{3} D_{1111}^{JG}(r) + \frac{4}{3} r^2 \int_r^\infty y^{-3} \left[D_{1111}^{JG}(y) + D_{\lambda\lambda\lambda\lambda}^{JG}(y) - 6 D_{11\gamma\gamma}^{JG}(y) \right] dy \\ & + \frac{4}{3} \int_0^r y^{-1} \left[D_{\lambda\lambda\lambda\lambda}^{JG}(y) - 3 D_{11\gamma\gamma}^{JG}(y) \right] dy. \end{aligned} \quad (\text{E9})$$

Comparison of (E9) and (62) shows that under the *JG* assumption our general formula (62) produces (E9), and therefore also (E8) and Batchelor's formula (E1). This further verifies our derivations.

Appendix F: Viscous Range and Inner Scales of $D_{ijkl}(\vec{r})$ and $D_{ij}(\vec{r})$

We present the viscous-range formulas for components of $D_{ijkl}(\vec{r})$ and $D_{ij}(\vec{r})$ in our chosen coordinate system, and use them to define inner scales parameterizing the transition between viscous and inertial ranges. Our purpose is to facilitate calculation of the mean-squared pressure gradient.

Recall our notation for derivatives $u_{ij} = \partial u_i / \partial x_j$. We define the following derivative moments:

$$d_{11} \equiv \langle (u_{11})^4 \rangle \quad (\text{F1})$$

$$d_{1\gamma} \equiv \langle (u_{11})^2 (u_{\gamma 1})^2 \rangle \quad (\text{F2})$$

$$d_{\lambda\lambda} \equiv \langle (u_{\lambda 1})^4 \rangle. \quad (\text{F3})$$

As before, λ and γ are 2 or 3, and can be equal or different. By Taylor series expansion of u_i , we have the viscous-range formulas,

$$D_{1111}(r) \simeq d_{11} r^4$$

$$D_{11\gamma\gamma}(r) \simeq d_{1\gamma} r^4$$

$$D_{\lambda\lambda\lambda\lambda}(r) \simeq d_{\lambda\lambda} r^4.$$

For r in the viscous range, we define the universal constants

$$\Lambda_{1\gamma} \equiv D_{11\gamma\gamma}(r) / D_{1111}(r) = d_{1\gamma} / d_{11}, \quad (\text{F4})$$

$$\Lambda_{\lambda\lambda} \equiv D_{\lambda\lambda\lambda\lambda}(r) / D_{1111}(r) = d_{\lambda\lambda} / d_{11}. \quad (\text{F5})$$

For r in the inertial range, we introduce the parameters C_{11} , $C_{1\gamma}$, and $C_{\lambda\lambda}$ such that

$$D_{1111}(r) = C_{11} \varepsilon^{4/3} r^{-q}$$

$$D_{11\gamma\gamma}(r) = C_{1\gamma} \varepsilon^{4/3} r^{-q} \quad (\text{F6})$$

$$D_{\lambda\lambda\lambda\lambda}(r) = C_{\lambda\lambda} \varepsilon^{4/3} r^{-q}.$$

By equating the viscous-range and inertial-range asymptotic formulas, we define inner scales $(l_{11}, l_{1\gamma}, l_{\lambda\lambda})$, one for each of the three functions,

$$\begin{aligned} d_{11} l_{11}^4 &= C_{11} \varepsilon^{4/3} l_{11}^q \\ d_{1\gamma} l_{1\gamma}^4 &= C_{1\gamma} \varepsilon^{4/3} l_{1\gamma}^q \\ d_{\lambda\lambda} l_{\lambda\lambda}^4 &= C_{\lambda\lambda} \varepsilon^{4/3} l_{\lambda\lambda}^q. \end{aligned} \tag{F7}$$

From (F7), we have

$$l_{11} = \left(C_{11} \varepsilon^{4/3} / d_{11} \right)^{1/(4-q)}, \tag{F8}$$

with analogous expressions for $l_{1\gamma}$ and $l_{\lambda\lambda}$. These inner scales are not equal, but we expect that their ratios are universal constants. The ratios of these inner scales are

$$\begin{aligned} l_{1\gamma} / l_{11} &= (\Lambda_{1\gamma} / H_{1\gamma})^{1/(q-4)} \\ l_{\lambda\lambda} / l_{11} &= (\Lambda_{\lambda\lambda} / H_{\lambda\lambda})^{1/(q-4)}, \end{aligned}$$

where, for r in the inertial range,

$$\begin{aligned} H_{1\gamma} &\equiv D_{111\gamma}(r) / D_{1111}(r) = C_{1\gamma} / C_{11} \\ H_{\lambda\lambda} &\equiv D_{\lambda\lambda\lambda\lambda}(r) / D_{1111}(r) = C_{\lambda\lambda} / C_{11}. \end{aligned}$$

We define the universal constants

$$h_{1\gamma} \equiv \Lambda_{1\gamma} \left(\frac{l_{1\gamma}}{l_{11}} \right)^2 = H_{1\gamma} \left(\frac{l_{1\gamma}}{l_{11}} \right)^{q-2} = \Lambda_{1\gamma}^{\frac{2-q}{4-q}} H_{1\gamma}^{\frac{2}{4-q}} \tag{F9}$$

$$h_{\lambda\lambda} \equiv \Lambda_{\lambda\lambda} \left(\frac{l_{\lambda\lambda}}{l_{11}} \right)^2 = H_{\lambda\lambda} \left(\frac{l_{\lambda\lambda}}{l_{11}} \right)^{q-2} = \Lambda_{\lambda\lambda}^{\frac{2-q}{4-q}} H_{\lambda\lambda}^{\frac{2}{4-q}}. \tag{F10}$$

We compare these inner scales with those of the second-order velocity structure function. First, we write the inertial-range formula in a manner that includes the intermittency effect:

$$D_{11}(r) = C \varepsilon^{2/3} r^g$$

$$D_{\lambda\lambda}(r) = C' \varepsilon^{2/3} r^g,$$

with $g = 2/3 + \mu/9$. The incompressibility condition (A25) gives

$$C' = \left(1 + \frac{g}{2}\right) C; \quad (\text{F11})$$

this gives the 3/4 factor if $g = 2/3$. For the viscous range, we have

$$D_{11}(r) = \langle (u_{11})^2 \rangle r^2 \equiv d_1 r^2$$

$$D_{\lambda\lambda}(r) = \langle (u_{\lambda 1})^2 \rangle r^2 \equiv d_\lambda r^2,$$

which defines the derivative moments d_1 and d_λ by implication. Incompressibility (A25) gives

$$d_\lambda = 2d_1. \quad (\text{F12})$$

Inner scales l_1 and l_λ are defined by equating viscous and inertial ranges at spacing equal to inner scale:

$$d_1 l_1^2 = C \varepsilon^{2/3} l_1^g$$

$$d_\lambda l_\lambda^2 = C' \varepsilon^{2/3} l_\lambda^g.$$

Thus,

$$l_1 = \left(C \varepsilon^{2/3} / d_1\right)^{1/(2-g)}$$

$$l_\lambda = \left(C' \varepsilon^{2/3} / d_\lambda\right)^{1/(2-g)}$$

$$\frac{l_1}{l_\lambda} = \left(\frac{4}{2+g}\right)^{1/(2-g)}$$

For $\mu = 0.25$, this ratio of inner scales is 1.3534, whereas for $\mu = 0$ (i.e., for $g = 2/3$), the ratio is 1.3554. Isotropy gives

$$\varepsilon = 15\nu d_1,$$

where ν is kinematic viscosity. Therefore,

$$\frac{l_1}{\eta} = (15C)^{1/(2-g)} \eta^{(3g-2)/(6-3g)},$$

where $\eta = (\nu^3/\varepsilon)^{1/4} = (\nu^2/15d_1)^{1/4}$ is the Kolmogorov microscale.

If $g = 2/3$ and C is the universal Kolmogorov constant ≈ 2 , then

$$\frac{l_1}{\eta} = 13.$$

However, with $\mu = 0.25$, the intermittency result is

$$\frac{l_1}{\eta} = 14 \left(\frac{C}{2} \right)^{36/47} \eta^{1/47}$$

Thus, l_1/η has weak dependence on both macrostructure and the microscale; from (F11), l_λ/η has the same dependence.

Comparing l_1 with l_{11} gives

$$\frac{l_{11}}{l_1} = \left(\frac{C^2 d_{11}}{C_{11} d_1^2} \right)^{-3/8} \left[\frac{\varepsilon^{4/3}}{C_{11} C^2 (15\nu)^4} \frac{d_{11}}{d_1^2} \right]^{\mu/32} \quad (\text{F13})$$

The second factor in (F13) is unique to intermittency theory; it has weak dependence on macrostructure and Reynolds number. The first factor also has this dependence arising from intermittency. We expect that all the inner scales are of similar value, so the first factor in (F13) is of the order of unity.

The moments d_{11} and $d_{\lambda\lambda}$ have been measured at modest Reynolds numbers and found to be nearly equal by Antonia et al. (1982b), who also found that $d_\lambda/d_1 \approx 1.6$, in contradiction to (F12). The streamwise velocity-derivative kurtosis,

$$K = d_{11}/d_1^2, \quad (\text{F14})$$

has been observed to vary from 4 to 40 as Reynolds number varies from laboratory to atmospheric values (Wyngaard and Tennekes, 1970). Consequently, from (F13), l_{11}/l_1 will vary considerably unless C_{11}/C^2 has a similar variation. Note that l_1 and l_{11} are not defined unless the Reynolds number is sufficiently large that an inertial range exists. The flatness factor, which is proportional to C_{11}/C^2 has been shown by Van Atta and Chen (1970) to be about 3 times larger in atmospheric data than in laboratory (lower Reynolds number) turbulence (see their Fig. 7).

Appendix G: Mean-Squared Pressure Gradient

As pointed out by Batchelor (1951), the mean-squared pressure gradient can be measured indirectly from measurements of turbulent diffusion. Using Batchelor's formula (54) for the mean-squared pressure gradient χ and the representation (C1) for $Q(r)$, we have

$$\chi = \frac{1}{6} \sum_{l=1}^3 \sum_{n=0}^4 W_{ln} \int_0^{\infty} dy y^{n-3} f_l^{(n)}(y). \quad (\text{G1})$$

We use integration by parts in Appendix I for the integrals in (G1); we obtain

$$\begin{aligned} \chi &= \sum_{l=1}^3 Y_{l4} \int_0^{\infty} dy y^{-3} f_l(y) \\ &= 4 \int_0^{\infty} dy y^{-3} [D_{1111}(y) + D_{\lambda\lambda\lambda\lambda}(y) - 6 D_{11\gamma\gamma}(y)] \\ &= 4 \int_0^{\infty} dy y^{-3} A_D(y), \end{aligned} \quad (\text{G2})$$

where Y_{l4} is defined in (D15) and given in Table C1. The sum in square brackets in (G2) vanishes as $y \rightarrow \infty$, which can be obtained by use of (26d) and (28).

We verify (G2) by substituting the JG relationships (B15) and (B16) to obtain

$$\begin{aligned} \chi^{JG} &= 4 \int_0^{\infty} dy y^{-3} \left\{ 3 [D_{11}(y)]^2 + 3 [D_{\lambda\lambda}(y)]^2 - 6 D_{11}(y) D_{\lambda\lambda}(y) \right\} \\ &= 4 \int_0^{\infty} dy y^{-3} 3 [D_{\lambda\lambda}(y) - D_{11}(y)]^2. \end{aligned}$$

Substituting the incompressibility condition (A25), we have

$$\chi^{JG} = 3 \int_0^{\infty} dy y^{-1} [D_{11}^{(1)}(y)]^2, \quad (\text{G3})$$

which, using (21), is seen to be the same as Batchelor's (1951) Eq. (5.7), thereby verifying (G2).

We define the quantity

$$H_\chi \equiv \frac{\int_0^\infty dy y^{-3} [D_{1111}(y) + D_{\lambda\lambda\lambda}(y) - 6 D_{11\gamma\gamma}(y)]}{\int_0^\infty dy y^{-3} D_{1111}(y)}. \quad (\text{G4a})$$

If we know H_χ as a function of the Reynolds number, then measurements of only u_1 give the denominator of H_χ , and we can then determine χ from (G2). For high Reynolds numbers, we define the universal constant

$$h_\chi \equiv 1 + h_{\lambda\lambda} - 6 h_{1\gamma}, \quad (\text{G4b})$$

where $h_{\lambda\lambda}$ and $h_{1\gamma}$ are universal constants given in Appendix F. We expect that H_χ becomes a universal constant at high Reynolds numbers, in which case we expect $H_\chi \approx h_\chi$.

The integral in (G2) depends on the viscous range and (for high Reynolds numbers) the inertial range. The shape of the structure-function components is poorly known near the transition between viscous and inertial ranges. The inner scale (Appendix F) parameterizing this transition is also poorly known. For large Reynolds numbers, we obtain an upper bound for χ by integrating the viscous-range formula from $y = 0$ to the inner scale and integrating the inertial-range formula for y beyond the inner scale. For instance, for $D_{1111}(r)$, using the results of Appendix F,

$$\begin{aligned} \int_0^\infty dy y^{-3} D_{1111}(y) &\leq d_{11} \int_0^{l_{11}} dy y + C_{11} \epsilon^{4/3} \int_{l_{11}}^\infty dy y^{q-3} \\ &= \frac{1}{2} d_{11} l_{11}^2 \left(1 + \frac{2}{2-q} \right) \end{aligned} \quad (\text{G5a})$$

$$= \frac{1}{2} C_{11} \epsilon^{4/3} l_{11}^{q-2} \left(1 + \frac{2}{2-q} \right) \quad (\text{G5b})$$

$$= 2\Gamma \left(1 - \frac{\mu}{4} \right). \quad (\text{G5c})$$

For (G5c), the quantity in brackets in (G5b) is approximated by $4 - \mu$, and

$$\Gamma \equiv d_{11} l_{11}^2 \equiv C_{11} \varepsilon^{4/3} l_{11}^{q-2} \geq D_{1111}(l_{11})/l_{11}^2. \quad (\text{G6})$$

We thus have from (G4a)

$$\chi \leq 8\Gamma \left(1 - \frac{\mu}{4}\right) H_\chi. \quad (\text{G7a})$$

Obtaining the other two integrals in (G2) in the same manner as (G5a,b) and substituting all three integrals in (G2) gives

$$\chi \leq 8\Gamma \left(1 - \frac{\mu}{4}\right) h_\chi. \quad (\text{G7b})$$

Hence, (G7a,b) give $H_\chi \approx h_\chi$ for large Reynolds numbers. The details of the transition between inertial and viscous ranges give, for large Reynolds numbers, $H_\chi = 1 + m_{\lambda\lambda} h_{\lambda\lambda} - 6 m_{1\gamma} h_{1\gamma}$, where the numerical coefficients $m_{\lambda\lambda}$ and $m_{1\gamma}$ are of order unity. Of course, $m_{\lambda\lambda} = m_{1\gamma} = 1$ gives (G4b).

Since $\chi > 0$, (G7a) gives the interesting bound

$$H_\chi > 0. \quad (\text{G8})$$

We expect that $l_{1\gamma}/l_{11}$ and $l_{\lambda\lambda}/l_{11}$ are of the order of unity, so (G8) is roughly $(1 - 6H_{1\gamma} + H_{\lambda\lambda}) \geq 0$, as well as $(1 - 6\Lambda_{1\gamma} + \Lambda_{\lambda\lambda}) \geq 0$, which say that in the inertial and viscous ranges, $D_{11\gamma\gamma}(r)$ must be less than roughly one-sixth of $[D_{1111}(r) + D_{\lambda\lambda\lambda\lambda}(r)]$. Such a quantitative prediction cannot be obtained from kinematics; it must be the result of using the Navier-Stokes equation as our first step.

Applying the same estimation method to (G3) as used in (G5a,b), we have the mean-squared pressure gradient in the *JG* approximation

$$\chi^{JG} \leq 8\Gamma^{JG} \left(1 + \frac{\mu}{6}\right), \quad (\text{G9})$$

where $\Gamma^{JG} \equiv d_1^2 l_1^2 = C^2 \varepsilon^{4/3} l_1^{2g-2}$. Take $\mu = 0$; then the ratio of (G7a) and (G9), considering that the same type of overestimate was used in both, gives

$$\frac{\chi}{\chi^{JG}} \propto H_\chi \left[\frac{C_{11}}{C^2} \left(\frac{d_{11}}{d_1^2} \right)^{1/3} \right]^{3/4}. \quad (G10)$$

The value of this ratio is unknown until H_χ is determined from experiment. However, from the discussion in Appendix F regarding velocity-derivative kurtosis and the flatness factor, the last factor in (G10) increases by a factor of about 4 as the Reynolds number varies from laboratory to atmospheric values. Consequently, $\chi^{JG} \propto \chi$ can at most occur for some particular value of Reynolds number.

We now consider the case of low Reynolds numbers, for which experimental data for χ has been obtained from dispersion experiments. Taylor's length scale λ_T , which parameterizes the initial decrease of the velocity correlation, is defined by

$$d_1 = \sigma_{11} / \lambda_T^2.$$

Batchelor (1951) similarly defines the pressure length scale λ_p by

$$\chi = \sigma_{11}^2 / \lambda_p^2.$$

Batchelor's (1951) Eq. (7.4) shows that $D_{11}(r)/\sigma_{11}$ is a function of only r/λ_T for low Reynolds numbers. His Fig. 1 shows that $D_{1111}(r)/[D_{11}(r)]^2$ is the same function of r/λ_T for various low Reynolds numbers. Consequently, using λ_T to scale the integration variable in (G4a), the denominator of H_χ is proportional to $(\sigma_{11}/\lambda_T)^2$. The proportionality constant could be obtained by numerical integration using Batchelor's (1951) Fig. 1 and Eq. (7.4). Consequently, for low Reynolds numbers, we have

$$H_\chi \propto (\lambda_p/\lambda_T)^{-2}.$$

Measured values of (λ_p/λ_T) , as summarized by Monin and Yaglom (1975), scatter from 0.4 to 1.0; Batchelor gives the value 0.81 on the basis of the *JG* assumption. Numerical simulation of the Navier-Stokes equation is a useful method of establishing the values of H_χ for low to moderate Reynolds numbers.

Appendix H: Pressure Variance

The case $m = 3$ in Appendix I gives the pressure variance from Batchelor's (1951) Eq. (2.16):

$$\begin{aligned}\sigma_p^2 &\equiv \frac{1}{\rho^2} \langle P^2 \rangle = -\frac{1}{2} I_3 = \\ &= -\frac{1}{12} \left[\sum_{r=1}^2 z_{r3} f_r(\infty) + W_{30} \int_0^\infty dy y^{-1} f_3(y) \right].\end{aligned}\tag{H1}$$

This applies to the second set of functions in Table C1; therefore,

$$\sigma_p^2 \equiv \frac{1}{6} D_{1111}(\infty) - D_{11\gamma\gamma}(\infty) + \frac{2}{3} \int_0^\infty dy y^{-1} [D_{\lambda\lambda\lambda\lambda}(y) - 3 D_{11\gamma\gamma}(y)].\tag{H2}$$

The integral in (H2) converges because of (28). Using (26c,d) and (28), we can write the first two terms in (H2) in a variety of ways, including $-D_{1111}(\infty)/6$. To check this result, we obtain from (H2), (B15), and (B16) the *JG* result

$$\left(\sigma_p^2\right)^{JG} \equiv \frac{1}{2} [D_{11}(\infty)]^2 - D_{11}(\infty) D_{\gamma\gamma}(\infty) + 2 \int_0^\infty dy y^{-1} \left\{ [D_{\lambda\lambda}(y)]^2 - D_{11}(y) D_{\gamma\gamma}(y) \right\}.\tag{H3}$$

Using the incompressibility condition (A25), we can write the integrand in (H3) as

$$\frac{1}{2} D_{11}^{(1)}(y) D_{\gamma\gamma}(y) = \frac{y}{4} [D_{11}^{(1)}(y)]^2 + \frac{1}{4} \left\{ [D_{11}(y)]^2 \right\}^{(1)}.\tag{H4}$$

The second term on the right side of (H4) can be immediately integrated to give $[D_{11}(\infty)]^2/4$; we then have

$$\left(\sigma_p^2\right)^{JG} \equiv [D_{11}(\infty)]^2 - D_{11}(\infty) D_{\gamma\gamma}(\infty) + \frac{1}{2} \int_0^\infty dy y [D_{11}^{(1)}(y)]^2.\tag{H5}$$

The first terms in (H5) vanish because $D_{11}(\infty)$ and $D_{\gamma\gamma}(\infty)$ are equal; they are both twice the velocity variance. The integral in (H5) is the same as Batchelor's (1951) Eq. (2.16). This verifies our result.

Appendix I: Definite Integrals of $r^m Q(r)$

We calculate the integrals

$$I_m \equiv \int_0^{\infty} dy y^m Q(y). \quad (I1)$$

According to Batchelor's (1951) Eqs. (2.12), (2.16), and (2.17), I_1 gives the mean-squared pressure gradient, I_3 gives the pressure variance, and $I_2 = I_4 = 0$, which will serve as a check of our results. We substitute the representation (C1) into (I1) and obtain

$$I_m = \frac{1}{6} \sum_{t=1}^3 \sum_{n=0}^4 W_{tn} G_{tnm}, \quad (I2)$$

where

$$G_{tnm} \equiv \int_0^{\infty} dy y^{m+n-4} f_t^{(n)}(y). \quad (I3)$$

We use the second set of functions in Table C1; this set includes $D_{11\gamma\gamma}(r) - (1/3)D_{\lambda\lambda\lambda\lambda}(r)$; by (28), this function set produces convergence of all integrals in (I2) for $m \leq 4$. For this function set, the integrals G_{10m} and G_{20m} do not exist in the sum (I2) because $W_{10} = 0 = W_{20}$; these integrals would be divergent for $m \geq 3$. We assume that $f_t(r)$ approaches $f_t(\infty)$ faster than does r^{-1} . Then the integrated terms vanish at the upper limit for $m < 3$. The integrated terms vanish at the lower limit for $m > 0$ because $f_t(r) \propto r^4$ in the viscous range. Our remaining integrals converge at the lower limit for $m > -1$, and they converge at the upper limit for $m \leq 4$. The integration by parts then produces the simple recurrence relation

$$G_{t(n+1)m} = (3-m-n)G_{tnm}. \quad (I4)$$

This result in (I2) gives

$$I_m = \frac{1}{6} \sum_{t=1}^3 Z_{tm} G_{t0m}, \quad \text{for } -1 < m < 3 \quad (I5)$$

$$I_m = \frac{1}{6} Z_{3m} G_{30m} + \frac{1}{6} \sum_{t=1}^2 z_{tm} G_{t1m}, \quad \text{for } 3 \leq m \leq 4. \quad (I6)$$

The distinction between (I5) and (I6) is that in (I6) the integration by parts was terminated so as not to generate the divergent integrals G_{10m} and G_{20m} from G_{11m} and G_{21m} . The coefficients in (I5) and (I6) are

$$z_{tm} \equiv W_{t1} + (2-m) \left\{ W_{t2} + (1-m) [W_{t3} - mW_{t4}] \right\},$$

$$\dot{Z}_{tm} \equiv W_{t0} + (3-m) z_{tm}.$$

From (D12), (D14), and Table C1, we see that $Z_{t2} = 2Y_{t3} = 0$ and $Z_{t4} = 6Y_{t1} = 0$, and $z_{t4} = W_{t0} - Z_{t4} = 0$. Consequently, (I5) gives $I_2 = 0$ and (I6) gives $I_4 = 0$; Batchelor (1951) derived that $I_2 = I_4 = 0$.

Appendix J: Compressibility Effects

Expression (69) is intractable and would be extremely difficult to evaluate experimentally. This is because it contains quantities that cancel to produce a very much smaller $M_p(r)$, as do the expressions by Obukhov and Yaglom (1951) relating $D_p(r)$ to $R_{ijkl}(\vec{r})$. Similar to the estimation of $D_p^{JG}(r)$, estimates of (69) can be obtained on the basis of the *SI* and *JG* assumptions using (50) to (53). Substituting (50) to (53) in (69) and rearranging terms gives

$$M_p^{JG}(r) = 4 \sigma_{\lambda\lambda} D_{11}(r) I(r) + 4 (\sigma_{11} - \sigma_{\lambda\lambda}) \left[D_{11}(r) + 4 r^2 \int_r^\infty y^{-3} [D_{\lambda\lambda}(y) - D_{11}(y)] dy + 2 \int_0^r y^{-1} D_{\lambda\lambda}(y) dy \right], \quad (J1)$$

wherein the fractional imbalance in incompressibility is defined by

$$I(r) \equiv \left\{ D_{11}(r) - 2 \int_0^r y^{-1} [D_{\lambda\lambda}(y) - D_{11}(y)] dy \right\} / D_{11}(r) \quad (J2a)$$

$$= \left\{ \int_0^r (D_{11}^{(1)}(y) - 2 y^{-1} [D_{\lambda\lambda}(y) - D_{11}(y)]) dy \right\} / D_{11}(r). \quad (J2b)$$

The numerator in curly brackets in (J2b) is the integral of the incompressibility condition (A25). Thus, $I(r) = 0$ for perfect incompressibility. For approximate incompressibility, $I(r)$ is small compared with unity because the denominator in (J2a) is the first term of the numerator. From (50) to (53), we see that $M_{ijkl}(\vec{r})$ is not locally isotropic (if anisotropy exists at large scales). Therefore, unlike $D_p(r)$, $M_p(r)$ is not locally isotropic. Hence, having assumed isotropy to obtain (3), we must assume isotropy (at all scales) for (69); in particular, we must assert that $\sigma_{11} = \sigma_{\lambda\lambda}$. Therefore, (J1) is

$$M_p^{JG}(r) = 4 \sigma_{11} D_{11}(r) I(r). \quad (J3)$$

Requiring

$$|M_p^{JG}(r)| \ll D_p(r) \quad (J4)$$

gives

$$|I(r)| \ll D_p(r)/[4\sigma_{11}D_{11}(r)]. \quad (J5)$$

From (J5), we have for the inertial range

$$|I(r)| \ll \frac{1}{2} F H_p(r/L_0)^{2/3+\mu/9}, \quad (J6)$$

where the flatness factor is given by

$$F \equiv D_{1111}(r)/[D_{11}(r)]^2 \propto r^{-4\mu/9}, \quad (J7)$$

and the outer scale L_0 is defined as in (23) and (24), although μ need not be zero. From (J6), we see that the constraint on incompressibility becomes more stringent as r/L_0 decreases. For r in the inertial range, smaller values of r/L_0 are accessible as the Reynolds number increases, so the stringency increases with the Reynolds number. For $r \approx L_0$, the right side of (J6) is of the order of unity, so the constraint is lax in the production range.

Viscous-range asymptotic formulas are the same for compressible and incompressible fluids because they are based on Taylor series expansions. Examples of such formulas are $D_{11}(r) = d_1 r^2$, $D_{\lambda\lambda}(r) = d_\lambda r^2$, and $D_p(r) = (1/3)\chi r^2$, where d_1 and d_λ are the derivative moments defined in Appendix F and χ is the mean-squared pressure gradient. Let $I(0)$ denote the limit of $I(r)$ as $r \rightarrow 0$; we have

$$I(0) = 2 - (d_\lambda/d_1). \quad (J8)$$

In the limit $r \rightarrow 0$, (J5) becomes

$$|I(0)| \ll \chi/12\sigma_{11}d_1. \quad (J9)$$

Using (G7a) and (F13) and neglecting the second factor in (F13), we have

$$|I(0)| \ll 30 H_\chi (d_{11}/d_1^2)^{1/4} (C_{11}/C^2)^{3/4} / R, \quad (J10)$$

where

$$R \equiv \sigma_{11}/\nu d_1^{1/2}$$

is the Reynolds number and ν is kinematic viscosity. As noted at the end of Appendix F, the numerator in (J10) increases with R . However, the denominator increases faster such that the right side of (J10) decreases by about a factor of 5 as R increases from 3×10^2 to 10^4 . Therefore, as the Reynolds number increases, the constraint on incompressibility becomes more stringent. If we use (F12) in (J8), perfect incompressibility requires that $I(0) = 0$. If the condition (J10) is violated, then our formulas for χ may be invalid.

Appendix K: The Acceleration Tensor

Monin and Yaglom (1975), Batchelor (1951), and Obukhov and Yaglom (1951) showed that the correlation tensor of the acceleration vector consists of two terms; one is acceleration by the pressure gradient and the other is acceleration by viscous friction. Batchelor (1951) and Monin and Yaglom (1975) showed that the former is much larger than the latter for very large Reynolds numbers, but the latter is by far the greater for very low Reynolds numbers. Entirely satisfactory expressions for the viscous acceleration term have been given by Monin and Yaglom (1975). These expressions involve many orders of differentiation of $D_{ij}(\vec{r})$. Here, we give new results for the pressure gradient acceleration tensor defined by

$$\begin{aligned} A_{ij}(\vec{r}) &\equiv \frac{1}{\rho^2} \langle P_{|i} P'_{|j} \rangle \\ &= - \frac{1}{\rho^2} \langle PP' \rangle_{|ij} = \frac{1}{2} D_p(r)_{|ij}. \end{aligned} \tag{K1}$$

The general formula (A23) for a second-order isotropic tensor applies. In our chosen coordinate system, we therefore have the components

$$A_{\alpha\beta}(r) = [A_{11}(r) - A_{22}(r)] \delta_{1\alpha} \delta_{1\beta} + A_{22}(r) \delta_{\alpha\beta}. \tag{K2}$$

This implies, of course, that $A_{33} = A_{22}$ and $A_{\alpha\beta} = 0$ if $\alpha \neq \beta$. Performing the covariant second-order derivative (K1) on $D_p(r)$ gives

$$A_{11}(r) = \frac{1}{2} D_p^{(2)}(r) \tag{K3a}$$

$$A_{22}(r) = \frac{1}{2r} D_p^{(1)}(r). \tag{K3b}$$

The curl of the gradient is identically zero, so $A_{ij}(\vec{r})$ must satisfy Eq. (12.70) by Monin and Yaglom (1975); namely

$$A_{11}(r) = A_{22}(r) + r A_{22}^{(1)}(r), \tag{K4}$$

which is seen to be satisfied by (K3a,b).

We rewrite (62) using (G2):

$$\begin{aligned}
D_p(r) &= \frac{\chi}{3} r^2 - \frac{1}{3} D_{iiii}(r) \\
&\quad - \frac{4}{3} r^2 \int_0^r y^{-3} [D_{iiii}(r) + D_{\lambda\lambda\lambda\lambda}(r) - 6D_{ii\gamma\gamma}(r)] dy \\
&\quad + \frac{4}{3} \int_0^r y^{-1} [D_{iiii}(r) - 3D_{ii\gamma\gamma}(r)] dy.
\end{aligned} \tag{K5}$$

Performing the first and second derivatives, we have

$$\begin{aligned}
A_{22}(r) &= \frac{\chi}{3} - \frac{1}{6r} D_{iiii}^{(1)}(r) - \frac{2}{3r^2} [D_{iiii}(r) - 3D_{ii\gamma\gamma}(r)] \\
&\quad - \frac{4}{3} \int_0^r y^{-3} [D_{iiii}(r) + D_{\lambda\lambda\lambda\lambda}(r) - 6D_{ii\gamma\gamma}(r)] dy
\end{aligned} \tag{K6a}$$

$$\begin{aligned}
A_{11}(r) &= \frac{\chi}{3} - \frac{1}{6} D_{iiii}^{(2)}(r) - \frac{2}{3r} [D_{iiii}(r) - 3D_{ii\gamma\gamma}(r)]^{(1)} \\
&\quad - \frac{2}{3r^2} [D_{iiii}(r) + 2D_{\lambda\lambda\lambda\lambda}(r) - 9D_{ii\gamma\gamma}(r)] \\
&\quad - \frac{4}{3} \int_0^r y^{-3} [D_{iiii}(y) + D_{\lambda\lambda\lambda\lambda}(y) - 6D_{ii\gamma\gamma}(y)] dy.
\end{aligned} \tag{K6b}$$

To obtain the viscous-range formulas, we can either substitute $D_{\alpha\alpha\beta\beta}(r) = d_{\alpha\beta} r^4$ in (K6a,b) or differentiate (55). In either method, we obtain

$$A_{22}(r) = \frac{\chi}{3} - 2 d_{11} h_Q r^2 + \dots \tag{K7a}$$

$$A_{11}(r) = \frac{\chi}{3} - 6 d_{11} h_Q r^2 + \dots \tag{K7b}$$

The universal constant h_Q is defined below (55). From (K7a,b) or (K6a,b), we see that $A_{ii}(0) = \chi$.

For the initial range, we substitute (G2) for χ into (K6a,b) and then use $D_{\alpha\alpha\beta\beta}(r) = C_{\alpha\beta} \varepsilon^{4/3} r^q$ to obtain

$$A_{22}(r) = \frac{1}{2} r^{-2} D_{1111}(r) q H_p \quad (\text{K8a})$$

$$A_{11}(r) = \frac{1}{2} r^{-2} D_{1111}(r) q(q-1) H_p. \quad (\text{K8b})$$

The universal constant H_p is given in (C13b). In Sec. 6, we show that H_p is difficult to evaluate because its value is sensitive to isotropy.

We can write (K7a,b) in the form

$$A_{\alpha\alpha}(r) = \frac{\chi}{3} \left(1 - \frac{r^2}{2\lambda_\alpha^2} + \dots \right), \quad (\text{K9})$$

which defines the length scale λ_α . For large Reynolds numbers, we obtain

$$\lambda_\alpha \approx \left[2 N_{11} H_\chi / (3 n_\alpha h_Q) \right]^{1/2} l_{11}, \quad (\text{K10})$$

where $n_1 = 6$ and $n_2 = 2$, and we introduce the dimensionless coefficient N_{11} chosen such that the denominator of (G4a) equals $N_{11} C_{11} \varepsilon^{4/3} l_{11}^{q-2}$.

We can write (K8a,b) in the form

$$A_{\alpha\alpha}(r) = \frac{\chi}{3} \left[\frac{n'_\alpha H_p}{n'_\alpha H_\chi} (l_{11}/r)^{2-q} \right], \quad (\text{K11})$$

where $n'_2 = 3q/8N_{11} \approx 1/3$ and $n'_1 = n'_2(q-1) \approx 1/9$. As r increases in the inertial range, (K11) shows that $A_{\alpha\alpha}(r)$ decreases approximately as $r^{-2/3}$.

There is no assurance that only the first term in (K6a) is positive, but this seems likely on the basis of the discussion of (G8). Therefore, it is likely that $A_{22}(r)$ is monotonically decreasing, in which case (K4) requires that $A_{11}(r)$ cross zero at some $r > \lambda_1$, thereby becoming negative; then $A_{11}(r)$ must cross zero again to become positive in the inertial range as required by (K11). As shown by Obukhov and Yaglom (1951), the JG -based theory predicts this behavior.