# Groundwave Height-Gain Functions Near a Shoreline 

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# Groundwave Height-Gain Functions Near a Shoreline 

R.M. Jones


#### Abstract

A saddlepoint approximation to a Kirchhoff integration over the surface of the Earth is used to derive formulas for the groundwave field for an elevated observer near a shoreline. It is shown that the transition from homogeneous groundwave propagation to mixed-path groundwave propagation occurs not at a vertical plane above the shoreline, but rather at an oblique surface tilted in the direction of propagation. Thus, close enough to the shoreline and high enough, the field over land (for sea-to-land propagation) will not be affected by the land beneath the observer, so that the field is represented in terms of sea-type groundwave modes (with the associated height-gain functions) even though the observer is above the land. This phenomenon is explained by interpreting groundwave modes as ground-reflected waves. There is a transition region (of several hundred kilometers horizontally for HF propagation) where diffractive corrections must be made because of the location of the shoreline.


## 1. INTRODUCTION

When groundwaves propagate across a discontinuity in ground surface impedance (as at a shoreline), there is coupling among groundwave modes. The formulas that give the coupling coefficients and the spatial variation of the groundwave field are well known (e.g., Hill and Wait, 1981, and the references they cite). Figure 1 shows the geometry for such mixed-path groundwave propagation.

The method used by Hill and Wait to calculate the spatial variation of the groundwave field is to calculate the groundwave signal strength for the transmitter and receiver on the Earth's surface and then extend the calculation to larger heights with height-gain functions appropriate to the ground directly below the transmitter or receiver. Although this method works well when the transmitter and receiver are far from the shoreline, it leads to an apparent discontinuity in the field above the shoreline (Jones, 1982). Although the discontinuity might disappear if enough groundwave modes were used, a comparison of calculations using 500 groundwave modes with calculations using 200 modes does not suggest that (Jones, 1982).

Possibly the problem is caused by use of approximations that are no longer valid for the higher-order groundwave modes or at large heights. Possibly the groundwave mode sum does not converge just beyond the shoreline for an elevated observer. There may be other reasons to explain the difficulties, but the result is that the usual methods do not seem to work well for an elevated observer near the shoreline.


Figure 1.-Geometry for mixed-path groundwave propagation.

A physical explanation (Jones, 1982) may be that the transition from one kind of groundwave mode representation to the next does not take place along a vertical plane above the shoreline, but instead along an oblique surface tilted toward the direction of propagation. Thus (for sea-to-land propagation), for an observer high enough, it is more appropriate to represent the field in terms of sea-type groundwave modes (with the associated height-gain functions) even if the observer is above the land. Section 3 explains the physical basis for this result.

The usual representation, in which the transition takes place at a vertical plane above the shoreline, may not be wrong, but may simply not be as useful as the one presented here.

Jones (1982) estimated the location of the transition surface on the basis of an ad-hoc argument. Here, I give a more rigorous estimate using a. Kirchhoff integral estimate of the field strength.

## 2. SUMMARY

For mixed-path groundwave propagation (as across a shoreline), the usual height-gain functions (e.g., Hill and Wait, 1981) are appropriate for an elevated observer when the observer is far from the shoreline. However, when the observer is close enough to the shoreline or high enough, the appropriate height-gain function is that for an observer above the sea for sea-to-land propagation even though the observer is above land. This effect is explained by realizing that a groundwave mode is a ground-reflected wave at nearly horizontal incidence. For sea-to-land propagation, an elevated observer sees the groundwave mode coming from the sea even though he may be above the land. There is a large transition region of several hundred kilometers near the shoreline where neither a simple land-type nor sea-type height-gain function is useful.


The complex reflection angle depends on the surface impedance of the ground, the radius of curvature of the ground, and the mode number.

Figure 2.--The boundary condition for groundwave modes.

## 3. RAY REPRESENTATION OF GROUNDWAVE MODES

To understand why it may not always be appropriate to calculate heightgain functions in the usual way near a shoreline, it is necessary to understand the ray representation of groundwave modes.

A groundwave mode can be represented in terms of a complex angular propagation constant $v$ (Jones, 1968a,b, 1982). That is, the field changes by an amount

$$
\begin{equation*}
E \propto \exp (-i \vee \theta) \tag{1}
\end{equation*}
$$

in propagating a great-circle angle $\theta$. The angular propagation constant $\nu$ is complex to give both amplitude and phase. The effective propagation constant in terms of linear distance d along the Earth's surface would be $v / a$, where a is the radius of the Earth. That groundwave modes propagate with nearly the same phase velocity as waves in free space is expressed by the fact that v/a is nearly the same as $k$, the free space wave number.

Mathematically, the representation of field strength in terms of groundwave modes comes from a residue expansion at the poles of the ground reflection coefficient (Wait, 1961; Berry, 1964; Jones, 1968a, b). Thus, groundwave modes are ground-reflected waves that reflect at an angle where the ground reflection coefficient is infinite. Physically, an infinite reflection coefficient means that it is possible for a reflected wave to satisfy the boundary conditions at the ground with no incident wave.

Thus, we can picture a groundwave as shown in Fig. 2. The angle $\beta_{s}$ of the groundwave from the horizontal is characteristic of the groundwave mode, and is given by

$$
\begin{equation*}
\cos \beta_{s}=\nu_{s} / k a \tag{2}
\end{equation*}
$$

where $\nu_{s}$ is the angular (complex) propagation constant of the groundwave mode, k is the free-space wavenumber, and $a$ is the Earth radius. Appendix A defines


Figure 3.--Geometry for a geometrical interpretation of height-gain functions.
these and other quantities more precisely. The parameter $v_{\mathrm{s}}$ is complex, to give both amplitude and phase variation of the groundwave mode with distance. Thus, the angle $\beta_{s}$ must also be complex because of (2). However, the angle $\beta_{s}$ is small, because $v_{s}$ is very close to $k a$. Thus, when we picture a groundwave mode as in Fig. 2, we must keep in mind that the groundwave mode propagates nearly horizontally.

Appendix $B$ gives an interpretation of groundwave height-gain functions using Fig. 3 that substantiates the above ray interpretation of groundwave modes.

Jones (1982) pointed out that this ray interpretation of groundwave modes implies that the amplitude of a groundwave mode should depend not on the properties of the ground below the receiver, but rather on the properties of the ground at the point from which the ray that represents the groundwave mode seems to be coming (that is, from the ground reflection point for the groundwave mode). If the effective ground reflection point is on the sea, then a height-gain function appropriate to the sea should be used to calculate the signal strength even if the receiver is above the land.

To estimate where the transition occurs is difficult, however, because the angle $\beta_{S}$ is complex, and therefore the position of the reflection point on the ground will also be complex. As an ad-hoc estimate, the transition was taken to occur when the real part of the position on the ground crossed the shoreline (Jones, 1982). The results seemed reasonable. However, here I use a Kirchhoff integration representation to calculate more rigorously where the transition occurs. As it turns out, the analysis is much more complicated than the simple picture suggested (Jones, 1982) and the transition regions are much larger.

## 4. KIRCHHOFF INTEGRAL REPRESENTATION FOR MIXED-PATH GROUNDWAVES

The arguments presented above suggest that the usual method for estimating height-gain functions might be inaccurate close to the shoreline. Those arguments, however, imply no inaccuracies for the field on the ground for mixed-path propagation. Thus, if we assume that the fields on the ground calculated by using the usual mixed-path groundwave propagation formulas (Hill and Wait, 1981) are accurate, then we can calculate the field above the ground using Kirchhoff integration in which we consider the integration to be carried out over the surface of the Earth.

To be complete, the Kirchhoff integration must include an integration over all surfaces that enclose the observer. Thus, we must also include an integration over a sphere at infinity. However, for our problem, we have no sources at infinity, so we can take that integral to be zero.

In addition, we must include a volume integral over all sources above the Earth, which includes the source dipole in our problem. However, we assume that the source dipole is so far away (and beyond the horizon) that the direct field from the source is negligible compared with the groundwave field. Thus, we neglect the direct contribution from the source.

Further, Stratton (1941, p. 468) shows that at a discontinuity in ground surface properties (as at a shoreline) there will be a line distribution of sources that contribute to the total observed field in addition to the Kirchhoff integral over the surface of the Earth. Tai (1972) shows that the Franz formula for surface integration includes directly the contribution of sources at the shoreline, whereas the Stratton-Chu formula (Stratton, 1941, pp. 464-468) does not.

For the present case of mixed-path propagation normal to the shoreline for a vertical electric dipole source, there is no contribution of these sources at the shoreline to the horizontal magnetic field at the observer. In addition, the sources at the shoreline contribute an electric field at the observer in a direction parallel to the straight line connecting the observer with the shoreline. Thus, within the approximations normally used, the contribution of these sources at the shoreline to the total vertical electric field at the observer would be small except very near the shoreline, so we neglect the line sources here.

Thus, the total significant field at the observer is given by Kirchhoff integration over the surface of the Earth. Figures 4 and 5 define the integration geometry. Figure 4 shows a spherical triangle on the Earth's surface connecting the source, the integration point, and the point directly below the observer. Figure 5 shows a vertical plane through the observer and integration point. Appendix $C$ shows that such an integration can give the horizontal magnetic field at the observer as

$$
\begin{equation*}
H=\sum_{\nu_{t}} \int_{C} \int_{-\pi}^{\pi} f\left(\nu_{t}, \nu, \alpha\right) \exp \left[-i P\left(\nu_{t}, \nu, \alpha\right)\right] d \alpha d \nu \tag{3}
\end{equation*}
$$



Figure 4.--Projection of the integration geometry on the Earth's surface. $\theta, \theta^{\prime}$, and $\gamma$ are central Earth angles.

Figure 5.--Geometry showing vertical plane through the observer and integration point.

where $C$ is a contour from the origin around the branch point $\nu=k a$ of $P$ and back to the origin,

$$
\begin{align*}
P\left(\nu_{t}, v, \alpha\right)= & -\nu_{t} \theta+\nu_{t} \cos ^{-1}(\cos \theta \cos \gamma+\sin \theta \sin \gamma \cos \alpha)+ \\
& \left(k^{2} r^{2}-v^{2}\right)^{1 / 2}-\left(k^{2} a^{2}-v^{2}\right)^{1 / 2},  \tag{4}\\
\gamma= & \cos ^{-1}(\nu / k r)-\cos ^{-1}(\nu / k a) \tag{5}
\end{align*}
$$

$\nu_{t}=\nu_{s}$ when the integration point is in the source region (on the sea),
$v_{t}^{t}=v_{r}^{s}$ when the integration point is in the observer region (on the land), $\theta$ is the great-circle angle between the source and the observer, a is the radius of the Earth, ( $r-a$ ) is the height of the observer, and $f$ is a function (defined in Appendix C) that is slowly varying except at the shoreline, where it is discontinuous because the boundary condition (and therefore, the field) is discontinuous there.

Equation (4) neglects azimuthal refraction that occurs when the groundwave mode is incident obliquely on the shoreline. However, when the great circle between the source and the observer is normal to the shoreline (the case considered here), the greatest contribution to the integral in (3) occurs for $\alpha=z e r o$, where there is no azimuthal refraction at the shoreline. In general, however, azimuthal refraction can be taken into account in a
straightforward manner. It is neglected here to simplify the equations, so that the main point of this report (finding the appropriate height-gain function near a shoreline) is not masked.

The parameters $v$ and $\alpha$ specify the position of the integration point on the surface of the Earth. $\alpha$ is the azimuthal angle of the point relative to the observer. $v$ is proportional to the cosine of the elevation angle of the integration point at the observer.

The physical interpretation of $P$ (apart from a constant term) is that it is the (complex) phase of the signal that propagates as a groundwave mode to the integration point and then as a straight-1ine ray to the observer. Thus, the integral in (3) could be interpreted as a path integral (e.g., Feynman and Hibbs, 1965).

If the field were specified exactly on the integration surface, then the integral (3) would give the field at an elevated observer exactly. Because the field specified on the ground is only approximate, the solution in (3) will also be only approximate. We consider here only the case in which distance from the source to the observer is much smaller than the size of the Earth. In that case, the actual limits used for the integrations in (3) are not important as long as they are large.

It will be noticed that the integration in (3) includes the part of the Earth's surface that is not in the line of sight of the observer. The straight-1ine part of the path passes through the Earth for those integration points. Of course such paths have no physical significance. The conductivity of the Earth is large enough that such paths would not contribute significantly to the field because of the great attenuation in passing through the Earth. However, such paths are treated in the integral in (3) as though the straight-line segment were in free space. Thus, no such attenuation would appear for those paths in (3). The justification for including such paths (and including them in that way) is given by Stratton (1941, p. 467). Stratton points out that the surface fields in the Kirchhoff integration are equivalent to a distribution of electric and magnetic sources on the surface. For the purpose of obtaining the fields outside the Earth, the original problem can be replaced by one in which we have these equivalent sources distributed over the surface of a sphere that represents the Earth. The inside of the Earth is replaced by free space, so that the total effect of the Earth on the propagation is then taken into account by the equivalent sources. Thus, the integration in (3) is justified, including those paths that appear to pass through the Earth.

If the exact field on the surface of the Earth were used in the Kirchhoff integration, then the integration would yield the correct field for an elevated observer. The actual field used in the integration here is a highfrequency approximation to the correct field (that is, the field for large ka). If correction terms were included in the surface field, these would contain higher powers of (ka) ${ }^{-1}$. These would in turn contribute terms to the integral that have higher powers of (ka) ${ }^{-1}$. We could neglect these terms to get the field for large ka. Thus, it is justified to neglect these higher order terms at the outset, and use only the high-frequency approximation for the field on the surface for the Kirchhoff integration.

The second term in (4) is the contribution to the phase of the groundwave mode as it propagates to the integration point. The final two terms in (4) give the contribution to the phase of the straight-line ray from the integration point to the observer.

## 5. SADDLEPOINT EVALUATION OF THE INTEGRALS

For the purposes of the present work, the main results come from the insight gained in looking at the saddlepoint evaluation of the integrals in (3) rather than from a more rigorous evaluation.

Mathematically, the saddlepoint occurs where $P$ is an extremum for variations of $\alpha$ and $v$. Thus, the integration point where $P$ is an extremum determines a path that satisfies a complex form of Fermat's principle.

Let us consider what path satisfies Fermat's principle. As we might expect from symmetry, for the azimuth integration, Fermat's principle determines that $\alpha$ is zero at the saddlepoint. This is a path that is in the vertical plane through the source and observer. As expected, this is the path that is the shortest as a function of $\alpha$. Appendix $D$ shows this in more detail.

Once the $\alpha$ integration has been done, we can represent the $v$ integration more explicitly as an integral in two parts as

$$
\begin{equation*}
H=\sum_{S} \int_{0}^{v^{\prime}} f_{3}(v) \exp \left[-i P\left(v_{S}, v\right)\right] d v+\sum_{r} \int_{v^{\prime}}^{0} f_{4}(v) \exp \left[-i P\left(v_{r}, v\right)\right] d \nu \tag{6}
\end{equation*}
$$

where $f_{3}$ and $f_{4}$ are slowly varying functions defined in Appendix $E$, and

$$
\begin{equation*}
P\left(\nu_{t}, v\right)=-\nu_{t} \gamma+\left(k^{2} r^{2}-v^{2}\right)^{1 / 2}-\left(k_{a}^{2}-v^{2}\right)^{1 / 2} \tag{7}
\end{equation*}
$$

Appendix $D$ shows that the saddlepoint condition leads to

$$
\begin{equation*}
v=\nu_{t}, \tag{8}
\end{equation*}
$$

in which there are three possibilities:
(1) The saddlepoint occurs in the source region far from the shoreline.
(2) The saddlepoint occurs in the observer region far from the shoreline.
(3) The saddlepoint occurs near the shoreline.

In the first case, we make a saddlepoint evaluation of the first integral in (6), neglect the effect of the upper limit, and neglect the second integral. Appendix $E$ makes a saddlepoint evaluation for that case, and shows that it leads to the same result as for homogeneous groundwave propagation, at least in the asymptotic limit, even when the observer is above the land. Thus this substantiates the physical picture proposed in Secs. 1 and 3.

In the second case, we make a saddlepoint evaluation of the second integral in (6), neglect the effect of the lower limit, and neglect the first integral. The same saddlepoint evaluation in Appendix E applies to this case also, and leads to the usual result for mixed-path groundwave propagation.

In the third case, neither integral in (6) can be neglected. The asymptotic evaluation of the integral is difficult in that case because it involves a saddlepoint near an endpoint and branch points near a saddlepoint. For the present purposes, it is not necessary to evaluate the integrals, but merely to point out that this third case is a transition case in which the evaluation of the field is more complicated.

For the present purposes, it is sufficient to indicate where the transition region occurs, and that is taken up in the next section.

## 6. TRANSITION REGIONS

We want to find out where the boundaries are that separate the three regions for the cases mentioned in Sec. 5. For the first and second case, we want the shoreline (which is one of the endpoints for each of the integrals in (6)) to be far from the saddlepoint. When that does not occur, we have the third case.

Figures 6-8 show these transition regions for a representative case. (The calculations are given in Appendix F.)

Figure 6 shows the transition regions for the first groundwave mode. The graph shows four regions. In the upper left corner, we have the region where, although the observer is above land, the correct height-gain function is for an observer above the sea.

If the observer is too low or too far from the shoreline, he will be in the next region, which is a transition region for sea-type groundwave modes. In this second region (which includes the horizon), sea-type groundwave modes rather than land-type groundwave modes still apply, but a diffractive correction must be added. In this case, the diffractive correction is needed not because the endpoint is too close to the saddlepoint, but because the endpoint is on the steepest ascent side of the stationary phase path through the saddlepoint rather than on the steepest descent side. Under other circumstances, the diffractive correction might be caused by the endpoint's being too close to the saddlepoint (in which we would say that the endpoint is within the first Fresnel zone of the saddlepoint). The correction increases as the observer moves farther from the shoreline or closer to the ground. If the observer is too far from the shoreline or too close to the ground, it is no longer useful to represent the field in terms of sea-type groundwave modes. Figure 6 shows the limit.

Beyond that limit, Fig. 6 shows a small region where neither land- nor sea-type groundwave modes give a good representation of the field. Such a gap does not always occur. For land-to-sea propagation, for example, in this same small region one could represent the field at the observer in terms of either land- or sea-type groundwave modes, although the diffraction correction in either case would be large.

Finally, as the observer moves even farther away from the shoreline or closer to the ground, he encounters the region where it is more appropriate to represent the field by land-type groundwave modes. This region is a transition region, however, because a diffractive correction is needed. The farther away the observer is from the shoreline, the smaller this correction becomes, and it approaches zero for very large distances. All of the cases I examined would require a diffractive correction to the land-type groundwave modes at all distances from the shoreline, although the correction is probably negligible for reasonably large distances.

Figure 6 seems to imply that even on the ground diffractive corrections are needed close to the shoreline. That contradicts a basic assumption here that the usual representation of the groundwave field is correct on the ground. This apparent contradiction needs to be investigated further.

Figure 7 shows the transition regions for several groundwave modes from 1 to 10. The higher order sea-type groundwave modes need a diffractive correction closer to the shoreline than do the lower order modes (at least for larger observer heights). The transition from sea-type to land-type groundwave modes occurs farther from the shoreline for higher order modes.

Close to the ground (less than a dimensionless distance $y$ of about 1 , or 200 m for the case in Figs. 6 and 7), it seems that a groundwave mode representation always needs a diffractive correction for mixed-path propagation. For the case in Fig. 7, a diffractive correction is not needed for sea-type groundwave modes when the distance from the shoreline to the horizon is greater than about 1 in dimensionless units ( $x$ ), or about 50 km .

Figure 8 shows an expansion of the lower left corner of Fig. 7. Only in the upper left corner of Fig. 8 is a sea-type groundwave mode representation valid without diffractive corrections.

## x, Dimensionless Distance of Observer from Shoreline



Distance of Observer from Shoreline, km

Figure 6.--Transition regions for groundwave mode 1. The source is on the sea (to the left of the plot) very far away. For an observer in the upper left corner of the plot, the correct height-gain function is that of a groundwave mode over the sea, even though the observer is above land. In the center region (which includes the horizon), sea-type height-gain functions are still appropriate, but it is necessary to make diffractive corrections to the field. In the region to the right, landtype height-gain functions are appropriate, but diffractive corrections are necessary. In terms of the physical units ( km and m ), the vertical scale is expanded by a factor of about 250. Thus, the boundary lines that appear nearly vertical are really nearly horizontal. The calculations here correspond to a radio frequency of 30 MHz , ground conductivity of the sea of $4 \mathrm{mho} / \mathrm{m}$, a dielectric constant for the sea of 80 , ground conductivity for the land of $0.01 \mathrm{mho} / \mathrm{m}$, and a dielertric constant for the land of 15 . Changing the wave frequency changes the scaling between physical units and nondimensional units on the coordinates [through equations (A1) and (A5)].


Distance of Observer from Shoreline, km

Figure 7.--Transition regions for several groundwave modes (see Fig. 6). The groundwave mode number makes a negligible difference in determining the boundary between the pure sea-type groundwaves in the upper left corner and the transition region (where diffractive corrections must be applied). The boundary between sea-type groundwaves and land-type groundwaves is farther from the shoreline for the higher order modes, although the diffractive corrections for either type of groundwave are probably large near the boundary.


Figure 8.--Transition regions for several groundwave modes; an expansion of the lower left corner of Fig. 7 (see Figs. 6 and 7 for a more complete description). In terms of physical units ( $k m$ and $m$ ), the vertical scale is expanded by a factor of about 20. The transition boundary between pure sea-type groundwaves and sea-type groundwaves requiring diffractive corrections does not depend monotonically on mode number. Below about 200 m , the dependence of the transition boundary on mode number is no longer negligible.

## 7. ACKNOWLEDGMENTS

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## Appendix A. Notation and Auxiliary Equations

$$
\begin{align*}
& \mathrm{x}=(\mathrm{ka} / 2)^{1 / 3} \mathrm{~d} / \mathrm{a}  \tag{Al}\\
& \mathrm{x}_{1}=(\mathrm{ka} / 2)^{1 / 3} \mathrm{~d}_{1} / \mathrm{a}  \tag{A2}\\
& x_{2}=(k a / 2)^{1 / 3} d_{2} / a  \tag{A3}\\
& x^{\prime}=(k a / 2)^{1 / 3} \mathrm{~d}^{1 / a}  \tag{A4}\\
& y=(2 / k a)^{1 / 3} k h  \tag{A5}\\
& q=-i(k a / 2)^{1 / 3} \Delta  \tag{A6}\\
& \Delta=Z_{s} / Z_{0}  \tag{A7}\\
& q_{1}=-i(k a / 2)^{1 / 3} \Delta_{1}  \tag{A8}\\
& q_{2}=-i(k a / 2)^{1 / 3} \Delta_{2}  \tag{A9}\\
& w_{1}{ }^{\prime}\left(t_{s}\right)=q w_{1}\left(t_{s}\right)  \tag{A10}\\
& w_{1}(t) \equiv \pi^{1 / 2}[B i(t)-i A i(t)]  \tag{All}\\
& w_{1}{ }^{\prime}\left(t_{s}\right)=q_{1} w_{1}\left(t_{s}\right)  \tag{A12}\\
& w_{1}{ }^{\prime}\left(t_{r}\right)=q_{2} w_{1}\left(t_{r}\right)  \tag{Al3}\\
& v_{s}=k a+(k a / 2)^{1 / 3} t_{s}  \tag{A14}\\
& v_{r}=k a+(k a / 2)^{1 / 3} t_{r}  \tag{A15}\\
& v_{t}=k a+(k a / 2)^{1 / 3} t_{t} \tag{A16}
\end{align*}
$$

$$
\begin{gather*}
W(x, q) \equiv \sum_{s=1}^{\infty} W\left(x, q, t_{s}\right)  \tag{A17}\\
W\left(x, q, t_{s}\right)=(\pi x / i)^{1 / 2} \exp \left(-i x t_{s}\right) /\left(t_{s}-q^{2}\right)  \tag{A18}\\
W^{\prime}\left(x, q_{1}, q_{2}\right)=\sum_{s=1}^{\infty} \sum_{r=1}^{\infty} W^{\prime}\left(x, q_{1}, q_{2}, t_{s}, t_{r}\right)  \tag{A19}\\
W^{\prime}\left(x, q_{1}, q_{2}, t_{s}, t_{r}\right)=(\pi x / i)^{1 / 2} \frac{q_{2}-q_{1}}{t_{r}-t_{s}} \frac{\exp \left(-i x_{1} t_{s}\right)}{t_{s}-q_{1}^{2}} \frac{\exp \left(-i x_{2} t_{r}\right)}{t_{r}-q_{2}^{2}}  \tag{A20}\\
H_{0}(d) \equiv \frac{i \omega}{c} \operatorname{Ids} \exp (-i K d) /(2 \pi d) . \tag{A21}
\end{gather*}
$$

## Appendix B. Geometrical Interpretation of Height-Gain Functions

The spatial variation of the field strength of a groundwave mode is given by Jones (1982) as

$$
\begin{equation*}
E(\theta, h) \propto \exp (-i v \theta) w_{1}\left(t_{s}-y\right) / w_{1}\left(t_{s}\right) \tag{BI}
\end{equation*}
$$

where $\theta$ is the central-Earth angle between the source and the observer, and $h$ is the height of the observer above the ground. The normalized height parameter $y$ is defined approximately by

$$
\begin{equation*}
2 / 3\left(y-t_{s}\right)^{3 / 2} \approx\left(k^{2} r^{2}-v^{2}\right)^{1 / 2}-v \cos ^{-1}(\nu / k r) \tag{B2}
\end{equation*}
$$

and

$$
\begin{equation*}
2 / 3\left(-t_{s}\right)^{3 / 2}=\left(k^{2} a^{2}-v^{2}\right)^{1 / 2} v \cos ^{-1}(v / k a) \tag{B3}
\end{equation*}
$$

where

$$
\begin{equation*}
r=a+h \tag{B4}
\end{equation*}
$$

a is the radius of the Earth, $k$ is the free-space wave number, $w_{1}$ is the Airy function defined in Appendix $A$, and $v$ is the (complex) angular propagation constant of the ground wave mode (Jones, 1968b; 1982), which is determined from $t_{s}$ by (Al4).

The first factor in (B1) gives the horizontal variation of the groundwave mode. The second factor is called the height-gain function, and gives the variation of the signal strength as a function of height.

It is easiest to see a geometrical interpretation for the height-gain function for large heights. For large heights, y as determined by (B2) is large and nearly real positive. Under those conditions, an asymptotic approximation for the Airy function

$$
\begin{equation*}
w_{1}\left(t_{s}-y\right) \approx \exp \left[-i \pi / 4-12 / 3\left(y-t_{s}\right)^{3 / 2}\right] /\left(y-t_{s}\right)^{1 / 4} \tag{B5}
\end{equation*}
$$

is valid (Wait, 1961; Abramowitz and Stegun, 1964, p. 488; Jones, 1968a, Appendix D). Using the parameters defined in Fig. 3, we can rewrite (B2) to give

$$
\begin{equation*}
2 / 3\left(y-t_{s}\right)^{3 / 2}=k\left(\ell+\ell_{t}\right)-v\left(\theta-\theta^{\prime}+\theta_{t}\right) \tag{B6}
\end{equation*}
$$

and (B3) to give

$$
\begin{equation*}
2 / 3\left(-t_{s}\right)^{3 / 2}=k \ell_{t}-v \theta_{t} \tag{B7}
\end{equation*}
$$

Substituting (B5), (B6), and (B7) into (B1) gives

$$
\begin{equation*}
E(\theta, h) \propto \frac{\exp \left(-i v \theta^{\prime}-i k \ell\right)}{\left(y-t_{s}\right)^{1 / 4}} \frac{\exp \left[-i \pi / 4-i 2 / 3\left(-t_{s}\right)^{3 / 2}\right]}{w_{1}\left(t_{s}\right)} . \tag{B8}
\end{equation*}
$$

The second factor on the right of (B8) is a constant for a given groundwave mode. We can interpret the first factor in (B8) using Fig. 3. The phase corresponds to a wave that propagates as a groundwave for a central-Earth angle $\theta^{\prime}$ (from the source to the integration point), then as a free-space wave for a straight-1ine distance $\ell$ (from the integration point to the observer). From the point of view of the observer, the groundwave mode appears as a wave that comes from the ground such that the wave normal direction of the wave makes an angle $\theta_{t}$ with the horizontal when the wave leaves the ground. The angle $\theta$ is (of course) complex, and is given (from Fig. 3) by (Gl). When $h$ is small enough, (B2) can be further approximated, as can (B3) (e.g., Wait, 1961; Hill and Wait, 1981), but then the above geometrical interpretation is hidden.

## Appendix C. Kirchhoff Integral for Mixed-Path Propagation

If the solution for the electric and magnetic field is known over a closed surface, then the field at any point within the volume bounded by that surface can be found by a Kirchhoff integration over that surface. Such a solution is exact if the field specified on the surface is correct. If there are sources within the volume, their contribution must be included.

In the present case, it is appropriate to perform the Kirchhoff integration over the surface of the Earth, because we assume that the standard formulas for mjxed-path propagation are accurate for an observer who is not elevated. To isolate the effect of an elevated observer from other effects, we assume that the source is on the ground and far enough away from the shoreline that we can represent the source entirely by the groundwave modes it excites. The surface of integration for the Kirchhoff integration will be the surface of the Earth plus a surface at infinity. We assume that the latter surface makes no contribution, so we consider only the contribution from the integration over the surface of the Earth. The field at the surface of the Earth is represented by groundwave modes excited by the source. We use the usual mixed-path groundwave propagation formulas to represent these.

The source is a vertical dipole, so the groundwave field will consist of a horizontal magnetic field (normal to the path of propagation) and an electric field in the vertical plane of propagation and nearly vertical. It is usual to calculate the vertical electric fields in groundwave propagation, partly to give the signal observed by a vertical electric dipole. Here, I shall consider the horizontal magnetic field because the formulas are slightly simpler. Apart from a constant factor (the impedance of free space), the horizontal magnetic field is nearly the same as the vertical electric field. They differ only because of the slight tilt of the electric field, and this results in the vertical electric field's having an additional factor of about $v / k a$. It should be noted that in the usual approximate formulas for groundwave propagation, factors of $\nu / \mathrm{ka}$ are usually ignored in the solution. Thus, within the usual approximations, there is no difference between the vertical electric field and the horizontal magnetic field (apart from a constant factor).

The Stratton-Chu formula (Stratton, 1941, p. 467; Jackson, 1962, pp. 283285; Jackson, 1975, p. 433) gives the magnetic field at the observer (position $z$ ) as
where $Z_{0}$ is the impedance of free space, $\vec{n}^{\prime}$ is a unit vector normal to the surface of integration pointing into the volume containing the observer, $\vec{z}^{\prime}$ ' is the integration point of the surface, and

$$
\begin{equation*}
G=\exp (-i k \ell) / \ell \tag{C2}
\end{equation*}
$$

is the free-space Green's function, where

$$
\begin{equation*}
\ell=\left|\vec{z}-\vec{z}^{\prime}\right| \tag{C3}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{\nabla}^{\prime} G=(i k+1 / \ell) G \hat{\ell}, \tag{C4}
\end{equation*}
$$

where

$$
\begin{equation*}
\hat{\ell}=\left(\vec{z}-\vec{z}^{\prime}\right) / \ell \tag{C5}
\end{equation*}
$$

is a unit vector pointing from the integration point to the observer.
The general formula in (C1) can be simplified for the present case, in which the surface of integration is the Earth's surface and the magnetic field on the ground is horizontal and normal to the vertical plane through the integration point and the source. First, (Cl) can be written entirely in terms of the magnetic field by using the surface impedance (e.g., Jackson, 1975, p. 772)

$$
\begin{equation*}
\vec{n}^{\prime} \times \vec{E}=Z_{s} \vec{n}^{\prime} \times\left(\overrightarrow{n^{\prime}} \times \vec{H}\right) \tag{C6}
\end{equation*}
$$

where $Z_{s}$ is the surface impedance of the Earth. We also have

$$
\begin{gather*}
\vec{n}^{\prime} \times\left(\vec{n}^{\prime} \times \vec{H}\right)=-\vec{H},  \tag{C7}\\
\vec{n} \cdot \cdot \cdot \vec{H}=0, \tag{C8}
\end{gather*}
$$

and

$$
\begin{equation*}
\left(\overrightarrow{\mathrm{n}}^{\prime} \times \overrightarrow{\mathrm{H}}\right) \times \overrightarrow{\nabla^{\prime}} \mathrm{G}=(\mathrm{ik}+1 / \ell) \mathrm{G}\left(\overrightarrow{\mathrm{n}}^{\prime} \times \overrightarrow{\mathrm{H}}\right) \times \hat{\ell} . \tag{C9}
\end{equation*}
$$

Figure 4 shows the integration geometry. The source, integration point, and vertical projection of the observer on the Earth's surface are shown. The triangle shown is a spherical triangle on the Earth's surface, and the sides of the triangle are great circles. Remembering that the observer is elevated, we can use (C5) and Fig. 4 to give

$$
\begin{equation*}
\left(\vec{n}^{\prime} \times \vec{H}\right) \times \hat{\ell}=\left(\vec{n}^{\prime} \cdot \hat{\ell}\right) \vec{H}-H \sin \varepsilon \vec{n}^{\prime}\left|\hat{\ell}-\vec{n}^{\prime} \cdot \hat{\ell}\right| . \tag{C10}
\end{equation*}
$$

Substituting (C2) through (C10) in (C1) gives

$$
\begin{align*}
\overrightarrow{\mathrm{H}}(\vec{z}) & =\frac{1}{4 \pi} \int_{\mathrm{S}}\left[-i k \Delta+(i k+1 / \ell) \vec{n}^{\prime} \cdot \hat{\ell}\right] \mathrm{G} \vec{H}^{(\vec{z}}\left(\vec{z}^{\prime}\right) \mathrm{d} S^{\prime} \\
& -\frac{1}{4 \pi} \int_{\mathrm{S}}(i k+1 / \ell) \sin \varepsilon\left|\hat{\ell}-\overrightarrow{\mathrm{n}}^{\prime} \cdot \hat{\ell}\right| \overrightarrow{\mathrm{n}}^{\prime} \mathrm{GH}\left(\vec{z}^{\prime}\right) \mathrm{d} S^{\prime}, \tag{C11}
\end{align*}
$$

where

$$
\begin{equation*}
\Delta=\mathrm{Z}_{\mathrm{s}} / \mathrm{Z}_{0} . \tag{C12}
\end{equation*}
$$

We see that we have broken the total magnetic field into two components. The first integral in (Cll) gives a component normal to the vertical plane through the source and the integration point. The second integral in (C11) gives a vertical component. The second term would be zero if the angle $\alpha$ in Fig. 4 were zero because then the angle $\varepsilon$ would be $180^{\circ}$. When we make a saddlepoint approximation later to the integrals in (Cll), we shall get $\alpha$ equal to zero at the saddlepoint, but for now we keep both terms.

We need to factor the integrands in (C11) into a slowly varying part times a quickly varying exponential $\underset{\rightarrow}{\text { part. }}$ The exponential part of $\underset{\rightarrow}{\text { (Cll) }}$ ) comes from the exponential parts of $G$ and $H\left(z^{\prime}\right)$. The exponential part of $H\left(z^{\prime}\right)$ is different for each groundwave mode. Thus, to proceed further, it is necessary to separate (Cl1) into a sum of groundwave modes of the field at the integration point $z$ '. To do that, we define

$$
\vec{H}\left(\vec{z}^{\prime}\right) \equiv \sum_{t} \vec{H}_{t}\left(\vec{z}^{\prime}\right) \equiv\left\{\begin{array}{l}
\sum_{s} \vec{H}_{s}\left(\vec{z}^{\prime}\right) \text { when } z^{\prime} \text { is on the sea }  \tag{C13}\\
\sum_{r} \vec{H}_{r}\left(\vec{z}^{\prime}\right) \text { when } z^{\prime} \text { is on the land } .
\end{array}\right.
$$

Thus, when the integration point $z^{\prime}$ is on the sea, we express the field at the observer as a sum of sea-type groundwave modes. When the integration point $z^{\prime}$ is on the land, we express the field as a sum of land-type groundwave modes. To treat both cases in a uniform way, we express either type of groundwave modes as a sum over $t$, where $t$ is simply a generic groundwave mode subscript. Thus we also define

$$
t_{t}=\left\{\begin{array}{l}
t_{s} \text { for } z^{\prime} \text { on the sea }  \tag{C14}\\
t_{r} \text { for } z^{\prime} \text { on the land }
\end{array}\right.
$$

and

$$
\nu_{t}=\left\{\begin{array}{l}
\nu_{s} \text { for } z^{\prime} \text { on the sea }  \tag{C15}\\
v_{r} \text { for } z^{\prime} \text { on the land. }
\end{array}\right.
$$

We next factor the field at the integration point into a unit vector $\hat{H}\left(\vec{z}^{\prime}\right)$ (which is slowly varying) times a magnitude $H\left(\vec{z}^{\prime}\right)$.

$$
\begin{equation*}
\overrightarrow{\mathrm{H}}\left(\vec{z}^{\prime}\right)=\hat{H}\left(\vec{z}^{\prime}\right) \mathrm{H}\left(\vec{z}^{\prime}\right) \tag{C16}
\end{equation*}
$$

We now factor the magnitude in a standard way

$$
\begin{align*}
& H\left(z^{\prime}\right)=H_{0}\left(d^{\prime}\right) W\left(x^{\prime}, q_{1}\right) \text { if } z^{\prime} \text { is on the sea }  \tag{C17a}\\
& H\left(z^{\prime}\right)=H_{0}\left(d^{\prime}\right) W^{\prime}\left(x^{\prime}, q_{1}, q_{2}\right) \text { if } z^{\prime} \text { is on the land, } \tag{C17~b}
\end{align*}
$$

where $H_{0}\left(d^{\prime}\right)$ is the field that would be observed a distance $d^{\prime}$ away from the source if both the source and observer were above a flat perfect conductor, and is defined in Appendix $A$. The factors $W$ and $W^{\prime}$ are correction factors defined below in terms of a sum of groundwave modes.

$$
\begin{align*}
& W\left(x, q_{1}\right) \equiv \sum_{s=1}^{\infty} W\left(x, q_{1}, t_{s}\right)  \tag{C18}\\
& W^{\prime}\left(x, q_{1}, q_{2}\right) \equiv \sum_{s=1}^{\infty} \sum_{r=1}^{\infty} W^{\prime}\left(x, q_{1}, q_{2}, t_{s}, t_{r}\right) \tag{C19}
\end{align*}
$$

The functions $W$ with three arguments and $W^{\prime}$ with five arguments are defined in Appendix A. The mixed-path propagation formula (Cl7b) is valid only if the groundwave propagation is normal to the shoreline. Thus, formulas in (C18) and (C19) are valid only when the shoreline is a circle concentric with the source. To simplify the calculations here, I assume that to be the case. These calculations could be extended to the more arbitrary situation using the formulas of Wait (1963a,b, 1964), but the additional complexity would detract from the main point of the present development.

To explicitly reveal the exponential dependence of $H_{0}\left(d^{\prime}\right), W\left(x, q_{1}, t_{s}\right)$, and $W^{\prime}\left(x, q_{1}, q_{2}, t_{s}, t_{r}\right)$, we use the definitions to write

$$
\begin{align*}
& H_{0}\left(d^{\prime}\right)=H_{0}(d)\left(d / d^{\prime}\right) \exp \left(-i k d^{\prime}+i k d\right)  \tag{C20}\\
& W\left(x^{\prime}, q_{1}, t_{s}\right)=W\left(x, q_{1}, t_{s}\right)\left(x^{\prime} x\right)^{1 / 2} \exp \left(-i x^{\prime} t_{s}+i x t_{s}\right)  \tag{C21}\\
& W^{\prime}\left(x^{\prime}, q_{1}, q_{2}, t_{s}, t_{r}\right)=W^{\prime}\left(x, q_{1}, q_{2}, t_{d}, t_{r}\right)\left(x^{\prime} / s\right)^{1 / 2} \exp \left(-i x^{\prime} t_{r}+i x t_{r}\right) \tag{C22}
\end{align*}
$$

We can express the above relations in a more uniform way if we define

$$
U \equiv\left\{\begin{array}{l}
W\left(x, q_{1}, t_{s}\right) \text { for } z^{\prime} \text { on the sea }  \tag{C23}\\
\sum_{s} W^{\prime}\left(x, q_{1}, q_{2}, t_{s}, t_{r}\right) \text { for } z^{\prime} \text { on the land. }
\end{array}\right.
$$

If we now make all the appropriate substitutions into (C11), we get

$$
\begin{equation*}
\vec{H}_{t}(\vec{z})=\int_{C} \int_{-\pi}^{\pi} \overrightarrow{\mathbf{f}}\left(\nu_{t}, \nu, \alpha\right) \exp \left[-i P\left(\nu_{t}, \nu, \alpha\right)\right] d \alpha d \nu \tag{C24}
\end{equation*}
$$

where $C$ is a contour from zero that goes around the branch point at $\nu=k a$ and back to zero,

$$
\begin{equation*}
P\left(\nu_{t}, v, \alpha\right)=k \ell+v_{t} \theta^{\prime}-v_{t} \theta \tag{C25}
\end{equation*}
$$

$\mathrm{k} \ell$ is given by (HI),

$$
\begin{equation*}
\theta^{\prime}=\cos ^{-1}(\cos \theta \cos \gamma+\sin \theta \sin \gamma \cos \alpha), \tag{C26}
\end{equation*}
$$

$\gamma$ is given by (H2),

$$
\begin{aligned}
& \overrightarrow{\mathbf{E}}\left(\nu_{t}, \nu, \alpha\right)=\frac{H_{0}(d) d^{1 / 2}}{4 \pi} U\left\{\left[-i k \Delta=(i k+1 / \ell) \vec{n}^{\prime} \cdot \hat{l}\right] \hat{H}\left(\vec{z}^{\prime}\right)\right. \\
& \left.-(i k+1 / \ell) \sin \varepsilon\left|\hat{\ell}-\vec{n}^{\prime} \cdot \vec{\ell}\right| \vec{n}^{\prime}\right\} \ell^{-1} d^{\prime}-1 / 2 a^{2} \sin \gamma\left[\left(k^{2} a^{2}-\nu^{2}\right)^{-1 / 2}\left(k^{2} r^{2}-\nu^{2}\right)^{-1 / 2}\right],(C 27)
\end{aligned}
$$

and

$$
\begin{align*}
\mathrm{dS}^{\prime} & =a^{2} \sin \gamma \mathrm{~d} \alpha \mathrm{~d} \gamma \\
& =a^{2} \sin \gamma\left[\left(k^{2} a^{2}-v^{2}\right)^{-1 / 2}-\left(k^{2} r^{2}-v^{2}\right)^{-1 / 2}\right] d \alpha d \nu \tag{C28}
\end{align*}
$$

Equations (H1) and (H2) (see Appendix H) come from the geometry of a vertical plane through the observer and integration point in Fig. 5. Equation (C26) comes from the spherical triangle geometry in Fig. 4. Also, from Fig. 5 we have

$$
\begin{equation*}
\overrightarrow{\mathrm{n}}^{\prime} \cdot \hat{\ell}=\left[1-\left(\frac{\nu}{\mathrm{ka}}\right)^{2}\right]^{1 / 2} \tag{C29}
\end{equation*}
$$

## Appendix D. Fermat's Principle

The complex phase in the integral in (C24) is [from (C25)]

$$
\begin{equation*}
P\left(\nu_{t}, \nu, \alpha\right)=k \ell+v_{t} \theta^{\prime}-v_{t} \theta, \tag{Dl}
\end{equation*}
$$

where from (H1),

$$
\begin{align*}
& k \ell=\left(k^{2} r^{2}-v^{2}\right)^{1 / 2}-\left(k^{2} a^{2}-v^{2}\right)^{1 / 2},  \tag{D2}\\
& \theta^{\prime}=\cos ^{-1}(\cos \theta \cos \gamma+\sin \theta \sin \gamma \cos \alpha), \tag{D3}
\end{align*}
$$

and from (H2),

$$
\begin{equation*}
\gamma=\cos ^{-1} \frac{\nu}{\mathrm{kr}}-\cos ^{-1} \frac{\nu}{\mathrm{ka}} \tag{D4}
\end{equation*}
$$

We want to find the values of $v$ and $\alpha$ for which $P$ is stationary. First, we consider variations in the azimuth angle $\alpha$. We have

$$
\begin{equation*}
\frac{\partial P}{\partial \alpha}=\nu_{y} \sin \theta \sin \gamma \sin \alpha / \sin \theta^{\prime} \tag{D5}
\end{equation*}
$$

Setting (D5) equal to zero gives the stationary point as

$$
\begin{equation*}
\alpha=0, \tag{D6}
\end{equation*}
$$

which is clear from the azimuthal symmetry of the geometry. To make a saddlepoint approximation to the integral in (C24), it is necessary to find the second derivative of $P$, and evaluate it at the saddlepoint specified in (D6). This gives

$$
\begin{equation*}
\left.\frac{\partial^{2} \mathrm{P}}{\partial \alpha^{2}}\right|_{\alpha=0}=\nu_{t} \sin \theta \sin \gamma / \sin \theta^{\prime} \tag{D7}
\end{equation*}
$$

Substituting (D6) into (D3) gives

$$
\begin{equation*}
\theta^{\prime}=\theta-\gamma \tag{D8}
\end{equation*}
$$

Substituting (D4) and (D8) into (D1) gives

$$
\begin{align*}
& P\left(\nu_{t}, \nu\right) \equiv P\left(\nu_{t}, \nu, 0\right)=k \ell=\nu_{t} \gamma \\
& \quad=\left(k^{2} r^{2}-\nu^{2}\right)^{1 / 2}-\left(k^{2} a^{2}-\nu^{2}\right)^{1 / 2}-\nu_{t}\left(\cos ^{-1} \frac{\nu}{k r}-\cos ^{-1} \frac{\nu}{k a}\right) \tag{D9}
\end{align*}
$$

Next, it is necessary to find the value of $v$ where $P$ is stationary with respect to variations of $v$. We have

$$
\begin{equation*}
\frac{\partial P}{\partial \nu}=\left[\left(k^{2} a^{2}-\nu^{2}\right)^{-1 / 2}-\left(k^{2} r^{2}-\nu^{2}\right)^{-1 / 2}\right]\left(\nu-v_{t}\right) \tag{D10}
\end{equation*}
$$

Setting (D10) to zero gives

$$
\begin{equation*}
\nu=v_{t} \tag{Dll}
\end{equation*}
$$

for the saddlepoint. Thus, at the saddlepoint, the ray from the integration point to the observer is at the same angle as the ray used to give the geometrical interpretation of the groundwave mode. To make the saddlepoint approximation for the $v$ integration in (C24), it is necessary to find the second derivative of P with respect to $v$, and evaluate it at the saddlepoint (DIl). This gives

$$
\begin{equation*}
\left.\frac{\partial^{2} p}{\partial \nu^{2}}\right|_{\nu=v_{t}}=\left(k^{2} a^{2}-v_{t}^{2}\right)^{-1 / 2}-\left(k^{2} r^{2}-v_{t}^{2}\right)^{-1 / 2} \tag{D12}
\end{equation*}
$$

Substituting (Dl1) into (D9) gives

$$
\begin{equation*}
P\left(v_{t}, \nu_{t}\right)=\left(k^{2} r^{2}-v_{t}^{2}\right)^{1 / 2}-\left(k^{2} a^{2}-v_{t}^{2}\right)^{1 / 2}-v_{t}\left(\cos ^{-1} \frac{\nu_{t}}{k r}-\cos ^{-1} \frac{\nu_{t}}{k a}\right) \tag{D13}
\end{equation*}
$$

## Appendix E. Saddlepoint Approximations

Appendix $D$ shows that the saddlepoint for the $\alpha$ integration in (C24) is at $\alpha$ equal to zero. The saddlepoint approximation for the $\alpha$ integration in (C24) is

$$
\begin{equation*}
\vec{H}_{t}(\vec{z})=\int_{C} \overrightarrow{\mathbf{f}}\left(\nu_{t}, v, 0\right)\left(\frac{2 \pi}{i \partial^{2} P /\left.\partial \alpha^{2}\right|_{\alpha=0}}\right)^{1 / 2} \exp \left[-i P\left(\nu_{t}, v, 0\right)\right] d v \tag{E1}
\end{equation*}
$$

When $\alpha$ is zero, (C27) simplifies because then $\varepsilon$ is $180^{\circ}$ (see Fig. 4). Thus,

$$
\begin{gather*}
\dot{\mathbf{f}}\left(\nu_{t}, \nu, 0\right)=\frac{H_{0}(d) d^{1 / 2}}{4 \pi} U\left\{-i k \Delta+(i k+1 / \ell)\left[1-\left(\frac{\nu}{k a}\right)^{2}\right]^{1 / 2}\right\} \hat{H}\left(\vec{z}^{i}\right) \\
\ell^{-1} d^{\prime}-1 / 2 a^{2} \sin \gamma\left[\left(k^{2} a^{2}-\nu^{2}\right)^{-1 / 2}-\left(k^{2} r^{2}-\nu^{2}\right)^{-1 / 2}\right] . \tag{E2}
\end{gather*}
$$

We see now that the vector $f$ (which is proportional to the magnetic field at the observer) is horizontal and perpendicular to the vertical plane through the source and observer (and thus has constant direction). Thus, we can simplify things from here on by writing only the magnitude of $f$.

The integrand in (El) changes discontinuously when the integration point moves from the sea to the land. Now that the $\alpha$ integration is finished, it is easy to express that change more explicitly.

$$
\begin{equation*}
H_{t}(\vec{z})=\int_{0}^{\nu^{\prime}} f_{3}(\nu) \exp \left[-i P\left(\nu_{s}, v\right)\right] d \nu+\int_{v^{\prime}}^{0} f_{4}(v) \exp \left[-i P\left(\nu_{r}, v\right)\right] d \nu \tag{E3}
\end{equation*}
$$

where $P\left(\nu_{t}, v\right)$ is given in (D9), $v^{\prime}$ is determined from

$$
\begin{equation*}
\cos ^{-1} \frac{v^{\prime}}{k r}-\cos ^{-1} \frac{v^{\prime}}{k a}=\theta_{2}=d_{2} / a \tag{E4}
\end{equation*}
$$

$\mathrm{d}_{2}$ is the distance of the observer from the shoreline, and from (D7) and (E2),

$$
\begin{align*}
f_{3}(v)= & H_{0}(d) W\left(x, q_{1}, t_{s}\right)\left\{-i k \Delta_{1}+(i k+1 / \ell)\left[1-\left(\frac{\nu}{k a}\right)^{2}\right]^{1 / 2}\right\} \\
& \frac{k a}{2}\left[\frac{\theta}{\theta-\gamma} \frac{a \ell}{k r} \frac{\sin (\theta-\gamma)}{2 \pi i \sin \theta\left(k^{2} r^{2}-v^{2}\right)\left(k^{2} a^{2}-v^{2}\right)}\right]^{1 / 2} \tag{E5}
\end{align*}
$$

and

$$
\begin{align*}
f_{4}(\nu)= & H_{0}(d) \sum_{S} W^{\prime}\left(x, q_{1}, q_{2}, t_{s}, t_{r}\right)\left\{-i k \Delta_{2}+(i k+1 / \ell)\left[1-\left(\frac{\nu}{k a}\right)^{2}\right]^{1 / 2}\right\} \\
& \frac{k a}{2}\left[\frac{\theta}{\theta-\gamma} \frac{a \ell}{k r} \frac{\sin (\theta-\gamma)}{2 \pi i \sin \theta\left(k^{2} r^{2}-\nu^{2}\right)\left(k^{2} a^{2}-\nu^{2}\right)}\right]^{1 / 2}, \tag{E6}
\end{align*}
$$

where, from (H2),

$$
\begin{equation*}
\gamma=\cos ^{-1} \frac{\nu}{k r}-\cos ^{-1} \frac{\nu}{k a} \tag{E7}
\end{equation*}
$$

and, from (H3),
$a \sin \gamma=\ell \frac{\nu}{k r}$.
Appendix D shows that the saddlepoint for the first integral in (E3) is at

$$
\begin{equation*}
v=v_{s} . \tag{E9}
\end{equation*}
$$

The saddlepoint integration of the first integral in (E3) gives

$$
\begin{equation*}
H_{s}(\vec{z})=f_{s}\left(\nu_{s}\right)\left(\frac{2 \pi}{i \partial^{2} P /\left.\partial \nu^{2}\right|_{\nu=\nu_{s}}}\right)^{1 / 2} \exp \left[-i P\left(\nu_{s}, \nu_{s}\right)\right] \tag{E10}
\end{equation*}
$$

Substituting (E5), (D12), and (D13) into (E10) gives

$$
\begin{align*}
& H_{s}(\vec{z})=H_{0}(d) W\left(x, q_{1} t_{s}\right)\left[-i k \Delta_{1}+(i k+l / \ell)\left(1-\frac{v_{s}}{k a}\right)^{l / 2}\right] \frac{k a}{2} \\
& {\left[\frac{\theta}{\theta-\gamma} \frac{a \ell}{k r} \frac{\sin (\theta-\gamma)}{i \sin \theta\left(k^{2} r^{2}-v_{s}^{2}\right)\left(k^{2} a^{2}-v_{s}^{2}\right)}\right]^{1 / 2}\left\{-\frac{1}{i\left[\left(k^{2} a^{2}-v_{s}^{2}\right)^{-1 / 2}-\left(k^{2} r^{2}-v_{s}^{2}\right)^{-l / 2}\right]}\right\}^{1 / 2}} \\
& \exp \left[-i\left(k^{2} r^{2}-v_{s}^{2}\right)^{1 / 2}+i\left(k^{2} a^{2}-v_{s}^{2}\right)^{1 / 2}+i \nu_{s} \cos ^{-1} \frac{\nu_{s}}{k r}-i v_{s} \cos ^{-1} \frac{\nu_{s}}{k a}\right] \quad \text { (E11) } \tag{E11}
\end{align*}
$$

The formulas (C21) and (C22) assume that the distance from the source to the integration point is small enough that

$$
\begin{equation*}
\sin (\theta-\gamma) \approx \theta-\gamma \tag{E12}
\end{equation*}
$$

Thus, we would expect that (E11) would approximate the correct solution only within the validity of (E12). Similarly to (E12), we also have the approximation

$$
\begin{equation*}
\sin \theta \approx \theta \tag{E13}
\end{equation*}
$$

It is thus valid to use (E12) and (E13) in (Ell) because (Ell) already implicitly assumes those approximations. Using (E12), (E13), (H4), (H5), (G1), and (G2) in (E11) gives

$$
\begin{align*}
& H_{s}(z)=H_{0}(d) W\left(x, q_{1}, t_{s}\right)\left[-\Delta+\left(1+\frac{1}{i k \ell}\right) \sin \theta_{t}\right] \frac{a}{2 r}\left[\frac{1}{\sin \left(\gamma+\theta_{t}\right) \sin \theta_{t}}\right]^{1 / 2} \\
& \exp \left[\left(-i k^{2} r^{2}-v_{s}^{2}\right)^{1 / 2}+i\left(k^{2} a^{2}-v_{s}^{2}\right)^{1 / 2}+i v_{s} \cos ^{-1} \frac{\nu_{s}}{k r}-i v_{s} \cos ^{-1} \frac{\nu_{s}}{k a}\right] \tag{E14}
\end{align*}
$$

The first two factors in (E14) give the field on the ground directly below the observer. Thus, the rest of (E14) should give the height-gain function

$$
\begin{equation*}
w_{1}\left(t_{s}-y\right) / w_{1}\left(t_{s}\right) \tag{E15}
\end{equation*}
$$

from (B1). To see the extent to which it does, we use (A8) and (A12) to give

$$
\begin{equation*}
\Delta_{1}=i\left(\frac{2}{\mathrm{ka}}\right)^{1 / 3} \mathrm{q}_{1}=i\left(\frac{2}{\mathrm{ka}}\right)^{1 / 3} \mathrm{w}_{1}^{\prime}\left(\mathrm{t}_{\mathrm{s}}\right) / \mathrm{w}_{1}\left(\mathrm{t}_{\mathrm{s}}\right) \tag{E16}
\end{equation*}
$$

We also use (B2) and (B3) (substituting $\nu_{s}$ for $\nu$ ). In addition, when $\theta_{t}$ and $\gamma+\theta_{t}$ are small, we can approximate (B2) and (B3) by

$$
\begin{equation*}
\sin \left(\gamma+\theta_{t}\right) \approx\left(\frac{2}{k a}\right)^{1 / 2}\left(y-t_{s}\right)^{1 / 2} \tag{E17}
\end{equation*}
$$

and

$$
\begin{equation*}
\sin \theta_{t} \approx\left(\frac{2}{k a}\right)^{1 / 3}\left(-t_{s}\right)^{1 / 2} \tag{E18}
\end{equation*}
$$

Substituting (E16), (B2), (B3), (E17) and (E19) in (E14) gives

$$
\begin{align*}
& H_{s}(z)=H_{0}(d) W\left(x, q_{1}, t_{s}\right)\left[-i \frac{w_{1}^{\prime}\left(t_{s}\right)}{w_{1}\left(t_{s}\right)}+\left(1+\frac{1}{i k \ell}\right)\left(-t_{s}\right)^{1 / 2}\right] \frac{a}{2 r} \\
& \quad\left(-t_{s}\right)^{-1 / 4}\left(y-t_{s}\right)^{-1 / 4} \exp \left[-i \frac{2}{3}\left(y-t_{s}\right)^{3 / 2}\right] \exp \left[i \frac{2}{3}\left(-t_{s}\right)^{3 / 2}\right] . \tag{E19}
\end{align*}
$$

In the present comparison, we are considering a high-frequency (short wavelength) limit. Thus, we assume

$$
\begin{equation*}
\mathrm{k} \ell \gg 1 \tag{E20}
\end{equation*}
$$

In addition, we assume that the observer is close enough to the ground that

$$
\begin{equation*}
a / r \approx 1 . \tag{E21}
\end{equation*}
$$

In fact, the approximation (E21) is already implicitly in (C21) (and therefore also in (E19)).

If the observer is high enough, then the asymptotic approximation (B5) is valid. When $t_{s}$ is negative and large enough, the same asymptotic approximation is valid also for

$$
\begin{equation*}
w_{1}\left(t_{s}\right) \approx \exp \left[-i \pi / 4-i \frac{2}{3}\left(-t_{s}\right)^{3 / 2}\right] /\left(-t_{s}\right)^{1 / 4} \tag{E22}
\end{equation*}
$$

When (E22) is valid, then we also have

$$
\begin{equation*}
w_{1}^{\prime}\left(t_{s}\right) / w_{1}\left(t_{s}\right) \approx i\left(-t_{s}\right)^{1 / 2} \tag{E23}
\end{equation*}
$$

Substituting (E20) through (E23) and (B5) in (E19) gives

$$
\begin{equation*}
H_{s}(z)=H_{0}(d) W\left(x, q_{1}, t_{s}\right) w_{1}\left(t_{s}-y\right) / w_{1}\left(t_{s}\right) \tag{E24}
\end{equation*}
$$

the known correct formula for homogeneous groundwave propagation.
In fact, the asymptotic forms (E22) and (E23) are not valid when $t_{s}$ is too small. In addition, (B5) is not always valid. However, as far as we know, (E24) is generally valid. Why does it appear to be valid only under the condition of certain asymptotic assumptions? The answer is that the integrals in (E3) contain branch points. The condition that one of the branch points is far enough from the saddlepoint that we may consider the saddlepoint isolated from it is exactly the condition for the asymptotic form (B5) to be valid. The condition that the other branch point is isolated from the saddlepoint is the same as the condition that the asymptotic form (E23) be valid. When the saddlepoint is isolated from these two branch points, the saddlepoint evaluation (E10) that led to (E19) is valid. Thus, (E19) agrees with (E24) under
the conditions that (E19) is valid, but does not agree with (E24) when the derivation leading to (E19) is not valid.

When the upper limit of the first integral in (E3) is zero (closed contour), we have the situation of homogeneous groundwave propagation. For that case, the first integral in (E3) must give (E24). We also know that the saddlepoint should give the dominant contribution to the integral. Therefore, it is reasonable to assume that the first integral in (E3) is approximately equal to ( E 24 ) whenever the saddlepoint is isolated from the endpoint and the path of integration can be easily deformed to go through the saddlepoint.

On the other hand, when the lower limit of the second integral in (E3) is small enough, the saddlepoint falls in the interval of the second integral rather than the first. In that case, we neglect the first integral in (E3) and evaluate the second integral by the saddlepoint approximation. In the same way that we derived (E19), this leads to

$$
\begin{gather*}
H_{r}(z)=H_{0}(d) \sum_{S} W^{\prime}\left(x, q_{1}, q_{2}, t_{S}, t_{r}\right)\left[-i \frac{w_{1}^{\prime}\left(t_{r}\right)}{w_{1}\left(t_{r}\right)}+\left(1+\frac{1}{i k \ell}\right)\left(-t_{r}\right)^{1 / 2}\right] \frac{a}{2 r} \\
\left(-t_{r}\right)^{-1 / 4}\left(y-t_{r}\right)^{-1 / 4} \exp \left[-i \frac{2}{3}\left(y-t_{r}\right)^{3 / 2}\right] \exp \left[i \frac{2}{3}\left(-t_{r}\right)^{3 / 2}\right] . \tag{E25}
\end{gather*}
$$

Similarly, using the asymptotic forms (B5), (E22), and (E23) and the approximations (E20) and (E21) in (E25) gives

$$
\begin{equation*}
H_{r}(z)=H_{0}(d) \sum_{s} W^{\prime}\left(x, q_{1}, q_{2}, t_{s}, t_{r}\right) w_{1}\left(t_{r}-y\right) / w_{1}\left(t_{r}\right), \tag{E26}
\end{equation*}
$$

the known correct formula for mixed-path groundwave propagation. The same discussion of (E19) and (E24) applies to (E25) and (E26). Therefore, it is reasonable to assume that the second integral in (E3) is approximately equal to (E26) whenever the saddlepoint is isolated from the endpoint and the path of integration can be easily deformed to go through the saddlepoint.

## Appendix F. Paths of Integration in the Complex Plane

To obtain the regions shown in Figs. 6-8, we need to find out when the saddlepoints of the integrals in (E3) are isolated from the end points and branch points. It is inconvenient to investigate the paths of integration in the complex $v$ plane because the formula for the complex phase (D9) is too complicated. We start out by writing (D9) in the following form:

$$
\begin{align*}
P\left(\nu_{t}, v\right) & =\left(k^{2} r^{2}-v^{2}\right)^{1 / 2}-\left(k^{2} a^{2}-v^{2}\right)^{1 / 2}-v\left(\cos ^{-1} \frac{v}{k r}-\cos ^{-1} \frac{v}{k a}\right) \\
& -\left(\nu_{t}-v\right)\left(\cos ^{-1} \frac{\nu}{k r}-\cos ^{-1} \frac{\nu}{k a}\right) \tag{F1}
\end{align*}
$$

To find out when the saddlepoints of the integrals in (E3) are isolated from the endpoints, it is useful to change the variable of integration from $v$ to $t$ defined by

$$
\begin{equation*}
\frac{2}{3}(-t)^{3 / 2}=\left(k^{2} a^{2}-v^{2}\right)^{1 / 2}-\nu \cos ^{-1} \frac{v}{k a} \tag{F2}
\end{equation*}
$$

We know that the saddlepoints of the integrals in (E3) are located at $v=$ $\nu_{t}$. Also, we are interested in determining when the endpoints of the integrals are near the saddlepoints and when the branch points are near the saddlepoints. Thus, it is allowable to make some approximations that are valid near the saddlepoint. The following approximate relations follow from (F2) for

$$
\begin{align*}
& \nu \approx k a \approx k r  \tag{F3}\\
& \frac{2}{3}(y-t)^{3 / 2} \approx\left(k^{2} r^{2}-\nu^{2}\right)^{1 / 2}-\nu \cos ^{-1} \frac{\nu}{k r}  \tag{F4}\\
& \cos ^{-1} \frac{\nu}{k a} \approx\left(\frac{2}{k a}\right)^{1 / 3}(-t)^{1 / 2}  \tag{F5}\\
& \cos ^{-1} \frac{\nu}{k r} \approx\left(\frac{2}{k a}\right)^{1 / 3}(y-t)^{1 / 2}  \tag{F6}\\
& \nu \approx k a+\left(\frac{k a}{2}\right)^{1 / 3} t \tag{F7}
\end{align*}
$$

In addition, we have, from (Al6)

$$
\begin{equation*}
\nu_{t}=k a+\left(\frac{k a}{2}\right)^{1 / 3} t_{t} . \tag{F8}
\end{equation*}
$$

Substituting (F2) through (F8) in (F1) gives

$$
\begin{equation*}
P(t) \equiv P\left(\nu_{t}, v\right)=\frac{2}{3}(y-t)^{3 / 2}-\frac{2}{3}(-t)^{3 / 2}-\left(t_{t}-t\right)\left((y-t)^{1 / 2}-(-t)^{1 / 2}\right) \tag{F9}
\end{equation*}
$$

To finish changing integration variables in (E3) we use the additional approximate relations

$$
\begin{align*}
& {\left[1-\left(\frac{v}{\mathrm{ka}}\right)^{2}\right]^{1 / 2} \approx\left(\frac{2}{\mathrm{ka}}\right)^{1 / 3}(-t)^{1 / 2}}  \tag{F10}\\
& {\left[1-\left(\frac{v}{\mathrm{kr}}\right)^{2}\right]^{1 / 2} \approx\left(\frac{2}{\mathrm{ka}}\right)^{1 / 3}(y-t)^{1 / 2},} \tag{Fll}
\end{align*}
$$

and take the derivative of (F2) to give

$$
\begin{equation*}
(-t)^{1 / 2} d t=\cos ^{-1} \frac{v}{k a} d v \tag{F12}
\end{equation*}
$$

Finally, substituting (E12), (E13), (E20), (A8), (A9), (F5), (F6), and (F9) through (F12) into (E3) through (E6) gives

$$
\begin{equation*}
H_{t}(\vec{z})=\int_{-\infty}^{t^{\prime}} f_{5}(t) \exp [-i P(t)] d t+\int_{t^{\prime}}^{-\infty} f_{6}(t) \exp [-i P(t)] d t \tag{F13}
\end{equation*}
$$

where $P(t)$ is given by (F9), $t^{\prime}$ is determined from

$$
\begin{align*}
& \left(y-t^{\prime}\right)^{1 / 2}-\left(-t^{\prime}\right)^{1 / 2}=\left(\frac{k a}{2}\right)^{1 / 3} \theta_{2}=x_{2}  \tag{F14}\\
& f_{5}(t)=\frac{-E_{0}(d) W\left(x, q_{1}, t_{s}\right)}{4(\pi i)^{1 / 2}} \frac{\left[q_{1}+i(-t)^{1 / 2}\right]\left[(y-t)^{1 / 2}-(-t)^{1 / 2}\right]^{1 / 2}}{(y-t)^{1 / 2}(-t)^{1 / 2}} \tag{F15}
\end{align*}
$$

and

$$
\begin{equation*}
f_{6}(t)=\frac{-E_{0}(d) \sum_{s} W^{\prime}\left(x, q_{1}, q_{2}, t_{s}, t_{r}\right)}{4(\pi i)^{1 / 2}} \frac{\left[q_{2}+i(-t)^{1 / 2}\right]\left[(y-t)^{1 / 2}-(-t)^{1 / 2}\right]^{1 / 2}}{(y-t)^{1 / 2}(-t)^{1 / 2}} \tag{F16}
\end{equation*}
$$

Both integrals in (F13) have the same form. There are branch points at $t=0$ and $t=y$, and the saddlepoint is at $t=t$. The location of the saddlepoint can be seen by taking the derivative of (F9):

$$
\begin{equation*}
\frac{d p}{d t}=\frac{1}{2}\left(t_{t}-t\right)\left[(y-t)^{-1 / 2}-(-t)^{-1 / 2}\right] \tag{F17}
\end{equation*}
$$

Taking the derivative of (F17) and evaluating it at the saddlepoint gives


Figure $\mathrm{F}-1 .-$ - Complex $t$ plane for integration for the sea-type groundwave mode 1. For homogeneous groundwave propagation, the original path of integration starts at $t=-\infty$ in the third quadrant (which corresponds to the antipode of the observer), goes along the negative real axis, around the branch point at the origin (which corresponds to the horizon of the observer), and along the negative real axis in the second quadrant to $t=-\infty$ (which corresponds to the point on the ground directly below the observer). There is a second branch point at $t=y=2$ ( $y=2$ corresponds to an observer height of about 400 m ). The saddlepoint occurs at ts=1.4133-1.733i. The location of the saddlepoint is determined by equation (A12), whose solution depends on the surface impedance of the sea, the free-space wavenumber, and the Earth radius. The original path of integration can be deformed to follow the stationary phase path through the saddlepoint. A saddlepoint evaluation of the integral leads to the usual groundwave mode formula including the usual height-gain function. For mixed-path propagation, the endpoint for the part of the integral over the sea moves from $-\infty$ to $t_{1}, t_{2}, t_{3}, t_{4}$, to $t_{5}$ as the observer moves over land farther away from the shoreline. When the observer is far enough away from the shoreline that the endpoint is between $t_{2}$ and $t_{4}$, diffractive corrections are necessary. When the observer is so far away from the shoreline that the endpoint is past $t_{4}$, it is no longer useful to deform the path of integration through the saddlepoint. The calculations here correspond to a radio frequency of 30 MHz , ground conductivity of the sea of $4 \mathrm{mho} / \mathrm{m}$, and a dielectric constant of 80 . Changing the wave frequency changes the scaling between physical units (height of the observer in $m$, and distance of the observer from the shoreline in km ) and the nondimensional units ( $y$ and the endpoint on the negative real axis) and will change the position of the saddlepoint.

$$
\begin{equation*}
\left.\frac{d^{2} p}{d t^{2}}\right|_{t=t}=-\frac{1}{2}\left[\left(y-t_{t}\right)^{-1 / 2}-\left(-t_{t}\right)^{-1 / 2}\right] \tag{F18}
\end{equation*}
$$

The path of integration in the $v$ plane is from the origin around the branch point at ka and back to the origin. Physically, this corresponds to integrating from the antipode ( $\nu=0$ ) to the horizon ( $\nu=k a$ ) and from the horizon to the point directly below the observer $(\nu=0)$. In the $t$ plane, the path of integration is from $-2(k a / 2)^{2 / 3}$ to the origin and back. In the $t$ plane, the branch point at the origin corresponds to the horizon. The first part of the path is from the antipode to the horizon; the second part is from the horizon to the point directly below the observer. For practical calculations, we can consider the path of integration to begin and end at $t=-\infty$.

The end point $t^{\prime}$ corresponds to the shoreline. The integrand is discontinuous at the shoreline las shown by splitting the integral into two parts in (Fl3)] because the field on the ground is discontinuous at the shoreline. If the shoreline is between the observer and the horizon, then the first integral in (Fl3) includes the part of the path around the branch point at the origin. If the shoreline is beyond the horizon, then the second integral includes the part with the branch point at the origin.

To find out when the saddlepoint approximation is valid, we need to evaluate $P-P_{s p}$ in the complex $t$ plane, where $P_{s p}$ is the value of $P$ at the saddlepoint. Figpure $\mathrm{F}-1$ shows such a plot.

There are branch points at the origin and at $t=y$, and the branch cuts are taken along the negative real axis and along the positive real axis from $t=y$ to $+\infty$.

For homogeneous groundwave propagation, we have only the first integral in (F13). The path of integration is along the negative real axis (below the branch line) to the origin, around the branch point at the origin, and back along the negative real axis. The beginning and end of the integration contour are so far away that they can be taken to be at $-\infty$.

The beginning of the integration contour is at the antipode of the observer. The branch point at the origin is at the horizon of the observer. The end of the integration contour is on the ground directly below the observer.

The saddlepoint is at $t=t_{s}$ in the fourth quadrant. It is possible to deform the path of integration to go through the saddlepoint along the stationary phase path shown in Fig. F-1. In so doing, the path does not cross any singularities. The contour is closed at sectors of infinity where $\exp (-i P)=0$. It is not possible to deform the path of integration to follow the steepest descent path because the steepest descent path is on the wrong side of the branch point at $t=y$.

For mixed-path propagation, we must consider the integrals in (F13), which have a finite endpoint. For the simple saddlepoint approximation to be valid, the endpoint in the integral must be far enough from the saddlepoint that

$$
\begin{equation*}
\left|P_{s 1}-P_{s p}\right|>1 \tag{F19}
\end{equation*}
$$

where $P_{s p}$ is the value of $P$ at the saddlepoint, and $P_{s 1}$ is the value of $P$ at the shoreline (the endpoint of the integration).

For the first integral in (F13), the integration is from $-\infty$ to $t^{\prime}$. Let us start with $t^{\prime}=-\infty$, and increase it to find out where the transition regions are.

For $t^{\prime}=-\infty$, we have homogeneous groundwave propagation in which both the observer and the source are above the sea. The contour can be deformed to follow the stationary phase path, and the saddlepoint approximation gives the usual homogeneous-path groundwave propagation formulas.

For $t^{\prime}$ finite (for example $t_{1}$ in Fig. $F-1$ ), it is still possible to deform the path of integration to follow the stationary phase path through the saddlepoint. The path of integration is then from the antipode to the horizon, and from the horizon to the shoreline. The upper limit $t^{\prime}$ of the integral can be increased until it reaches $t_{2}$, where it is on the stationary phase path. When the endpoint $t^{\prime}$ is larger (as at $t_{3}$ in Fig. 9), to get from the endpoint to the stationary phase path it is necessary to go along the path from $t_{3}$ to $t_{2}$, where

$$
\begin{equation*}
\left|\exp \left[-i\left(\mathrm{P}_{\mathrm{s} 1}-\mathrm{P}_{\mathrm{sp}}\right)\right]\right|>1 \tag{F20}
\end{equation*}
$$

Thus, although it is still possible to make a saddlepoint approximation to the integral, there is a correction term equal to the integral from the endpoint ( $t^{\prime}=t_{3}$ ) to the stationary phase path at $t_{2}$. The size of that correction depends on how far the stationary phase path is from the endpoint. Normally, it would be possible to evaluate the integral for an endpoint near the saddlepoint using error functions or Fresnel integrals, but the presence of the branch points makes that more difficult.

As the observer moves farther from the shoreline, the endpoint $t$ ' moves to the origin, around the origin, and along the negative real axis in the fourth quadrant. When the observer is so far from the shoreline that the endpoint is to the left of the path of steepest ascent at $t_{4}$, it no longer makes sense to deform the path of integration for the first integral in (F13) through the saddlepoint. Instead, it is more reasonable to simply integrate the first integral in (Fl3) along the negative real axis.

We now consider the second integral in (Fl3), which corresponds to integration over the part of the path on land. Again, we start with the observer on the shoreline. Then the endpoints of the second integral coincide, and there is no contribution from that mixed-path part of the integral, as expected.

As the observer moves back from the shoreline, the lower limit on the second integral in (F13) moves to a point such as $\mathrm{t}_{1}$ in Fig. F-2. The integration contour is along the negative real axis from $t_{1}$ to $-\infty$ in the second quadrant. We can neglect the integral because


Figure F-2.-Complex $t$ plane for integration for the land-type groundwave mode 1 (see Fig. $\mathrm{F}-1$ for a general description). The saddlepoint is at $t_{s}=1.1743-1.9955 i$. This figure differs from Figure $F-1$ only because of the location of the saddlepoint. For homogeneous groundwave propagation over land, we can consider the shoreline to be very far away. As the observer moves closer to the shoreline, the endpoint of the integral moves from $-\infty$ in the fourth quadrant to $t_{5}$ to $t_{4}$ and farther. When the observer is so close to the shoreline that the endpoint is past 44 (e.g., at $t_{3}$ ), it is no longer useful to deform the path of interration through the saddlepoint. The calculations here correspond to a radio frequency of 30 MHz , ground conductivity of the land of $0.01 \mathrm{mho} / \mathrm{m}$, and a dielectric constant of 15 .

$$
\begin{equation*}
\left|\exp \left[-i\left(P_{s l^{-P}}\right)\right]\right|<1 \tag{F21}
\end{equation*}
$$

along the path.
As the observer moves farther back from the shoreline, the endpoint moves to $t_{2}$, where it is on the stationary phase path. There is still no significant contribution to the second integral in (F13).

As the observer moves even farther back from the shoreline, the endpoint moves past the stationary phase line to a point such as that at $t_{3}$ in Fig. F -2. There is now a significant contribution to the integral from $\mathrm{t}_{3}$ to $\mathrm{t}_{2}$, which must be included in the calculation.

As the observer moves still farther back from the shoreline (beyond the horizon), the endpoint $t^{\prime}$ moves around the branch point at the origin to the point $t_{4}$ on the path of steepest ascent. If the observer moves farther from the shoreline than this (so that the endpoint of the integral is at $t_{5}$, for example) it is reasonable to deform the path of integration through the saddlepoint. It is necessary to add to the saddlepoint evaluation of the integral an integration from $t_{5}$ to the second line of steepest descent, however. This can be considered a sort of diffractive correction because the shoreline is within the first Fresnel zone for the ray that satisfies Fermat's principle.

No matter how far the observer moves back from the shoreline, however, it appears from Fig. F-2 that there will always be some correction because the second stationary phase line approaches the negative real axis, but does not intersect it. The correction seems to approach zero as the observer moves farther from the shoreline, however.

The points $t_{2}$ and $t_{4}$ in Figs. $F-1$ and $F-2$ thus seem to be boundaries of regions for the integrals in (F13). When the distance of the observer from the shoreline is such that the endpoint is above the branch line and to the left of $t_{2}$, we can represent the field as a sea-type groundwave with no diffractive corrections. When the endpoint is between $t_{2}$ and $t_{4}$, we can represent the field as a sea-type groundwave, but with diffractive corrections from both the integrals in (F13). When the endpoint is in the third quadrant to the left of $t_{4}$, we can represent the groundwave as a land-type groundwave, but with diffractive corrections from both the integrals in (Fl3).

Figure 6-8 show the boundaries of these regions. The lines to the left of the horizon (shown as a dashed line) show where $t^{\prime}=t_{2}$ for sea-type groundwave modes. The lines to the right of the horizon show where $t^{\prime}=t_{4}$ for both sea-type and land-type groundwave modes.

## Appendix G. Equations From the Geometry of Figure 3

$$
\begin{align*}
& \cos \theta_{t}=\frac{\nu}{k a}  \tag{G1}\\
& k \ell=\left(k^{2} r^{2}-v^{2}\right)^{1 / 2}-\left(k^{2} a^{2}-v^{2}\right)^{1 / 2}  \tag{G2}\\
& \theta^{\prime}=\theta-\gamma . \tag{G3}
\end{align*}
$$

## Appendix H. Equations From the Geometry of Figure 5

$$
\begin{align*}
& k \ell=\left(k^{2} r^{2}-\nu^{2}\right)^{1 / 2}-\left(k^{2} a^{2}-\nu^{2}\right)^{1 / 2}  \tag{H1}\\
& \gamma=\cos ^{-1} \frac{\nu}{k r}-\cos ^{-1} \frac{\nu}{k a}  \tag{H2}\\
& a \sin \gamma=\ell \cos \left(\gamma+\theta_{t}\right)=\frac{\ell \nu}{k r}  \tag{H3}\\
& \left(k^{2} r^{2}-\nu^{2}\right)^{1 / 2}=k r \sin \left(\gamma+\theta_{t}\right)  \tag{H4}\\
& \left(k^{2} a^{2}-\nu^{2}\right)^{1 / 2}=k a \sin \theta_{t} . \tag{H5}
\end{align*}
$$

